On Dynamic Inequalities of Hilbert-Type

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Abstract

In this paper, we present new weighted diamond alpha dynamic inequalities of Hilbert type on time scales. Our approach is based on the application of the reversed Hölder's inequality, the chain rule, and the mean inequality within the framework of diamond alpha calculus. Furthermore, we demonstrate that, as particular cases of our general results (for $\mathbb{T} = \mathbb{N}$ and $\mathbb{T} = \mathbb{R}$), one can obtain the reversed discrete and continuous forms of Hilbert-type inequalities.

1.   INTRODUCTION

David Hilbert proved Hilbert's double-series inequality without exact determination of the constant in his lectures (see [1]) and showed that if $\{a_k\}_{k=1}^{\infty}$ and $\{b_z\}_{z=1}^{\infty}$ are two real sequences such that $0 < \sum_{k=1}^{\infty} a_k^2 < \infty$ and $0 < \sum_{z=1}^{\infty} a_z^2 < \infty,$ then

$$\sum_{z=1}^{\infty}\sum_{k=1}^{\infty} \frac{a_k\, b_z}{k+z} \leq \pi \left(\sum_{k=1}^{\infty} a_k^2\right)^{\frac{1}{2}} \left(\sum_{z=1}^{\infty} b_z^2\right)^{\frac{1}{2}}. \tag{1.1}$$

In 1911, Schur [2] discovered the integral analogue of (1.1), which became known as the Hilbert integral inequality.

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy \le \pi \left(\int_0^\infty f^2(x)\,dx\right)^{\frac{1}{2}} \left(\int_0^\infty g^2(y)\,dy\right)^{\frac{1}{2}}, \tag{1.2}$$

for real functions f, g such that $0 < \int_0^\infty f^2(x)\,dx < \infty$, and $0 < \int_0^\infty y^2(y)\,dy < \infty$. The constant π in (1.1) and (1.2) is the best possible constant factor.

In 1925, by introducing a pair of conjugate exponents (p,q) $(p,q > 1$ with $1/p + 1/q = 1)$ Hardy [3] gave an extension of (1.1) as follows: If p, q > 1 and $a_k, b_z \ge 0$ are such that

$0 < \sum_{k=1}^\infty a_k^p < \infty$ and $0 < \sum_{z=1}^\infty b_z^q < \infty$, then

$$\sum_{z=1}^\infty \sum_{k=1}^\infty \frac{a_k b_z}{k+z} \le \frac{\pi}{\sin\frac{\pi}{p}} \left(\sum_{k=1}^\infty a_k^p\right)^{\frac{1}{p}} \left(\sum_{z=1}^\infty b_z^q\right)^{\frac{1}{q}}. \tag{1.3}$$

Hardy and Reisz [1] proved the equivalent integral analogue of (1.3).

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}d_x d_y \le \frac{\pi}{\sin\frac{\pi}{p}} \left(\int_0^\infty f^p(x)\,dx\right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y)\,dy\right)^{\frac{1}{q}}, \tag{1.4}$$

for nonnegative functions f and g such that $0 < \int_0^\infty f^p(x)\,dx < \infty$ and $0 < \int_0^\infty g^q(y)\,dy < \infty$. The constant factor $\pi / sin(\pi/p)$ in (1.3) and (1.4) is the best possible. In 1998, Pachpatte [4] gave a new inequality close to that of Hilbert: Let $a(m):\{0,1,2,\ldots,p\} \subset N \to R$ and $b(\vartheta):\{0,1,2,\ldots,q\} \subset N \to R$ with $a(0) = b(0) = 0$. Then

$$\sum_{m=1}^p \sum_{\vartheta=1}^q \frac{|a_m||b_\vartheta|}{m+\vartheta}$$

$$\le C(p,q)\left(\sum_{m=1}^p (p-m+1)|\nabla a_m|^2\right)^{\frac{1}{2}} x \left(\sum_{\vartheta=1}^q (q-\vartheta+1)|\nabla b_\vartheta|^2\right)^{\frac{1}{2}}, \tag{1.5}$$

where $\nabla_{a_m} = a_m - a_{m-1}$ and $c\,(p,\,q) = \frac{1}{2}\sqrt{pq}$.

In 2002, Kim et al. [5] proved that if $\lambda, \mu > 1$ and $a(m):\{0,1,2,\ldots,p\} \subset N \to R$ and $b(\vartheta):\{0,1,2,\ldots,q\} \subset N \to R$ with $a(0) = b(0) = 0$, then

$$\sum_{m=1}^p \sum_{\vartheta=1}^q \frac{|a_m||b_\vartheta|}{\mu m^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}} + \lambda\vartheta^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}} \le D^*(\lambda,\mu,p,q)\left(\sum_{m=1}^p (p-m+1)|\nabla a_m|^\lambda\right)^{\frac{1}{\lambda}}$$

$$x\left(\sum_{\vartheta=1}^q (q-\vartheta+1)|\nabla b_\vartheta|^\mu\right)^{\frac{1}{\mu}}, \tag{1.6}$$

where $\nabla a_m = a_m - a_{m-1}$ and $D^*(\lambda,\mu, p, q) = \frac{1}{\lambda+\mu} p^{\frac{\lambda-1}{\lambda}} q^{\frac{\mu-1}{\mu}}$.

Also, Kim et al [5] proved the continuous analogue of (1.6). Let $\lambda, \mu > 1$, and let f and g be real continuous functions on the intervals $(0, x)$ and $(0, y)$, respectively, with $f(0) = g(0) = 0$. Then

$$\int_0^x \int_0^y \frac{|f(m)||g(t)|}{\mu m^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}} + \lambda \vartheta^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}} dsdt$$

$$\leq M(\lambda,\mu,x,y)\left(\int_0^x (x-m)|f'(m)|^\lambda dm\right)^{\frac{1}{\lambda}} \left(\int_0^y (y-t)|g'(t)|^\mu dt\right)^{\frac{1}{\mu}}, \tag{1.7}$$

for $x, y \in (0,\infty)$, where $n(\lambda, \mu, x, y) = \frac{1}{\lambda+\mu} x^{\frac{\lambda-1}{\lambda}} y^{\frac{\mu-1}{\mu}}$.

In 2011, Chang-Jian et al. [6] generalized (1.5) as follows: Let $p_i > 1$ and $1/p_i + 1/p_i^* = 1$. Let $a_i(m_i)$ be real sequences defined for $m_i = 0, 1, 2, \ldots, c_i$ such that $a_i(0) = 0$, i = 1, 2, $\ldots$, n. Define the operator ∇ by $\nabla a_i(m_i) = a_i(m_i) - a_i(m_i - 1)$ for any function $a_i(m_i)$, i = 1, 2, $\ldots$, n. Then

$$\sum_{m_1=1}^{c_1} \cdots \sum_{m_n=1}^{c_n} \frac{\left(n - \sum_{i=1}^n \frac{1}{p_i}\right)^{n-\sum_{i=1}^n 1/p_i^*}}{\prod_{i=1}^n c_i^{1/p_i^*}} \cdot \frac{\prod_{i=1}^n |a_i(m_i)|}{\left(\sum_{i=1}^n m_i/p_i^*\right)^{\sum_{i=1}^n 1/p_i^*}}$$

$$\leqslant \prod_{i=1}^n \left(\sum_{m_i=1}^{c_i} (c_i - m_i + 1)|\nabla a_i(m_i)|^{p_i}\right)^{\frac{1}{p_i}}. \tag{1.8}$$

Also, the authors of [6] proved that if $h_i \geq 1$ and $p_i > 1$ are constants, $1/p_i + 1/p_i^* = 1$, and $f_i(m_i)$ are real-valued differentiable functions on $[0, x_i)$, where $x_i \in (0,\infty)$, such that $f_i(0) = 0$ for i = 1, 2, $\ldots$, n, then

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\left(n - \sum_{i=1}^n \frac{1}{p_i}\right)^{n-\sum_{i=1}^n \frac{1}{p_i}}}{\prod_{i=1}^n h_i x_i^{\frac{1}{p_i^*}}} \frac{\prod_{i=1}^n \left|f_i^{h_i}(m_i)\right|}{\left(\sum_{i=1}^n m_i/p_i^*\right)^{\sum_{i=1}^n 1/p_i^*}} d_{m_n} \cdots d_{m_1}$$

$$\leq \prod_{i=1}^n \left(\int_0^{x_i} \frac{(x_i - m_i)}{f_i^{h_i-1}(m_i)} \cdot f_i'(m_i) d_{m_i}\right)^{\frac{1}{p_i}}. \tag{1.9}$$

Over the past few decades, the theory of time scales has evolved as a powerful mathematical framework introduced to unify and generalize continuous and discrete calculus. A time scale, denoted by T, is defined as any nonempty closed subset of the real numbers R. This theory provides a unified approach to studying dynamic equations and inequalities that can model both continuous and discrete phenomena simultaneously.

The results in this paper contain the classical continuous and discrete inequalities as particular cases where T = R and T = N, respectively. In addition, these inequalities can be extended to the corresponding inequalities on various time scales such as T = h^N, h > 0, and T = q^N for q > 1. In recent years, many authors have contributed to the development of dynamic inequalities on time scales, exploring various generalizations and applications. For a detailed account of these studies, one may refer to [11-16].

While recent studies, such as those by Saied (2024), have explored reversed Hilbert-type inequalities specifically within the framework of nabla calculus, the present study extends this line of inquiry into the more comprehensive domain of diamond-alpha calculus.The primary objective of this paper is to derive new reversed forms of Hilbert-type dynamic inequalities within the setting of diamond-alpha calculus. To achieve this, we employ essential analytical tools such as integration by parts, the reversed Hölder's inequality on time scales, and the reverse mean inequality. These results provide a broader framework that unifies and extends known inequalities in both discrete and continuous cases.The aim of this paper is to determine assumptions for establishing some new reversed forms of inequalities (1.8) and (1.9) on time scales by establishing some new Hilbert-type inequalities on time scales nabla calculus. Our results will be proved by applying the integration by parts, reverse Hölder's inequality on time scales, and the reverse of mean inequality.

The organization of the paper as follows. In Sect. 2, we present some definitions, properties, and lemmas on time scales needed in Sect. 3, where we prove our results. These results, as particular cases where $\mathbb{T} = \mathbb{N}$ and $\mathbb{T} = \mathbb{R}$, give the reverse of inequalities (1.8) and (1.9), respectively.

2.   PRELIMINARIES

The concept of time scales was first introduced by Stefan Hilger (1988) in his Ph.D. thesis in order to unify continuous and discrete analysis into a single theoretical framework. His idea was to develop a calculus that works simultaneously for differential and difference equations. Later 2001, Bohner and Peterson [7] defined the time scale $\mathbb{T}$ as an arbitrary nonempty closed subset of the real numbers R. Also, they defined the backward jump operator as $\rho(\tau)$:= sup{m ∈ T : m < τ }. For any function f: $\mathbb{T} \to \mathbb{R}$, f^P (τ) denotes f (ρ(τ)). We define the time scale interval $[a, b]$ $_T$ by$[a, b]_T$:=[a,b] ∩ $\mathbb{T}$.

Definition 2.1 [8] A function λ: $\mathbb{T} \to$ R is left-dense continuous or continuous if it is constant at left-dense points in $\mathbb{T}$ and its right-sided limits exist at right-dense points in $\mathbb{T}$. The space of continuous functions is denoted by C ($\mathbb{T}$, $\mathbb{R}$).

Definition 2.2 [14] Let $\mathbb{T}$ be a time scale, $f: \mathbb{T} \to \mathbb{R}$ a function, and $t \in \mathbb{T}_k^k$. In this case, for $\alpha \in [0,1]$, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all m in a neighborhood U of the point t, with $\mu_{tm} = \sigma(t) - $ m and $\nu_{tm} = \rho(t) - m$, the inequality

$$|\alpha[f^{\sigma}(t) - f(m)]\nu_{tm} + (1 - \alpha)[f^{\rho}(t) - f(m)]\mu_{tm} - f^{\diamond\propto}(t)\mu_{tm}\nu_{tm}| < \varepsilon|\mu_{tm}\nu_{tm}|$$

holds. If this inequality is satisfied, then the expression $f^{\diamond\propto}(t)$ is called the diamond alpha derivative of the function f on $\mathbb{T}_k^k$.

Lemma 2.1 [14] Let $f: \mathbb{R} \to \mathbb{R}$ be a differentiable function and let $g: \mathbb{T} \to \mathbb{R}$ be a function that is diamond alpha differentiable at $t \in \mathbb{T}_k^k$. Then the composition $f \circ g$ is diamond alpha differentiable at t, and its diamond alpha derivative is given by:

$$(f \circ g)^{\diamond\alpha}(t) = f'(g(t)) \cdot g^{\diamond\alpha} \quad , \qquad (2.1)$$

where f is the usual derivative of f and $g^{\diamond\alpha}(t)$ is the diamond alpha derivative of g at point t.

Definition 2.3 [9] Let $a, t \in \mathbb{T}$ and $g: \mathbb{T} \to \mathbb{R}$ be a function. In this case, for $\alpha \in [0,1]$, the $\diamond_\alpha$ integral is defined as

$$\int_a^t g(\tau) \diamond_\alpha \tau = \alpha \int_a^t g(\tau)\Delta\tau + (1 - \alpha) \int_a^t g(\tau)\nabla\tau$$

The $\diamond_\alpha$ integral is a linear combination of the Δ (delta) and ∇ (nabla) integrals.

In general, for $t \in \mathbb{T}$, the equality

$$\left(\int_a^t g(\tau) \diamond_\alpha \tau \right)^{\diamond\alpha} = f$$

does not hold.

Teorem 2.1 [9] Let $a, b, t \in \mathbb{T}$ and $c \in \mathbb{R}, y, b: \mathbb{T} \to \mathbb{R}$. In this case

(i)
$$\int_a^t [y(\tau) + b(\tau)] \diamond_\alpha \tau = \int_a^t y(\tau) \diamond_\alpha \tau + \int_a^t b(\tau) \diamond_\alpha \tau$$

(ii)
$$\int_a^t wf(\tau) \diamond_\alpha \tau = w \int_a^t f(\tau) \diamond_\alpha \tau$$

(iii)
$$\int_a^t y(\tau) \diamond_\alpha \tau = - \int_t^a y(\tau) \diamond_\alpha \tau$$

(iv) $\int_a^t y(\tau) \diamond_\alpha \tau = \int_a^j y(\tau) \diamond_\alpha \tau + \int_h^t y(\tau) \diamond_\alpha$

(v) $\int_a^a y(\tau) \diamond_\alpha \tau = 0$

Lemma 2.2 [8] Let $a, b \in \mathbb{T}$, and let $y, g: \mathbb{T} \to \mathbb{R}$ be continuous functions. Then, the integration by parts formula with respect to the diamond$-\alpha$ integral is given by:

$$\int_a^b y(t)g^{\diamond\alpha}(t) \diamond_\alpha t = [y(t)g(t)]_a^b - \int_a^b y^{\diamond\alpha}(t)g(t) \diamond_\alpha t \qquad (2.2)$$

provided the involved integrals and derivatives exist.

Lemma 2.3 [10] Let $f, g: \mathbb{T} \to \mathbb{F}$ be positive functions that are diamond alpha integrable on the interval $[a, b] \subset \mathbb{T}$, where $\mathbb{T}$ is a time scale and $\alpha \in [0,1]$. If $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the reverse Hölder inequality in the context of diamond alpha integrals is given by:

$$\int_a^b f(t)g(t) \diamond_\alpha t \geq \left(\int_a^b f^p(t) \diamond_\alpha t \right)^{\frac{1}{p}} \left(\int_a^b g^q(t) \diamond_\alpha t \right)^{\frac{1}{q}} \tag{2.3}$$

provided that the right-hand side integrals exist and are finite.

Lemma 2.4 [10] Let $a_i, b_i \in \mathbb{T}$, and let either $\psi_i \in C\left([a_i, b_i]_\mathbb{T}, (-\infty, 0]\right)$ be nonincreasing functions or $\psi_i \in C([a_i, b_i]_\mathbb{T}, [0, \infty))$ be nondecreasing functions with $\psi_i(a_i) = 0, i = 1,2, \dots, n$. Then

$$\int_{a_i}^{\xi_i} \left| \psi_i^{\diamond_\alpha}(t_i) \right| \diamond_\alpha t_i = |\psi_i(\xi_i)|, \xi_i \in [a_i, b_i]_\mathbb{T} \tag{2.4}$$

Lemma 2.5 [1] If $\alpha_i, \beta_i > 0$ for $i = 1,2, \dots, n$, then

$$\prod_{i=1}^n \alpha_i^{\beta_i} \leq \frac{\left(\sum_{i=1}^n \alpha_i \beta_i \right)^{\sum_{i=1}^n \beta_i}}{\left(\sum_{i=1}^n \beta_i \right)^{\sum_{i=1}^n \beta_i}} \tag{2.5}$$

Lemma 2.6 [10] Let $q_i < 0$ with $1/p_i + 1/q_i = 1$, and let $s_i > 0, i = 1,2, \dots, n$. Then

$$\prod_{i=1}^n s_i^{1/q_i} \geq \frac{\left(\sum_{i=1}^n \frac{s_i}{q_i} \right)^{\sum_{i=1}^n \frac{1}{q_i}}}{\left(n - \sum_{i=1}^n \frac{1}{p_i} \right)^{\left(n - \sum_{i=1}^n \frac{1}{p_i} \right)}} \tag{2.6}$$

3. MAIN RESULT

Theorem 3.1 Let $a_i, b_i \in \mathbb{T}, 0 < p_i < 1, q_i < 0$, such that $\frac{1}{p_i} + \frac{1}{q_i} = 1$. Let ω_i weight functions. Let either $\psi_i \in C([a_i, b_i]_\mathbb{T}, (-\infty, \infty)$ be nonincreasing or $\psi_i \in C([a_i, b_i]_\mathbb{T}, (-\infty, \infty))$ be nondecreasing functions with $\psi_i(a_i) = 0, i = 1,2, \dots, n$. Then

$$\int_{a_n}^{b_n} \dots \int_{a_1}^{b_1} \frac{\left(n - \sum_{i=1}^n \frac{1}{p_i} \right)^{n - \sum_{i=1}^n \frac{1}{p_i}}}{\left(\sum_{i=1}^n \frac{\xi_i - a_i}{q_i} \right)^{\sum_{i=1}^n \frac{1}{q_i}}} \prod_{i=1}^n (b_i - a_i)^{1/q_i} w_i(\xi_i) |\psi_i(\xi_i)| \diamond_\alpha \xi_1 \dots \diamond_\alpha \xi_n$$

$$\geq \prod_{i=1}^n \left(\int_{a_i}^{b_i} w_i(\xi_i) | \diamond_\alpha \psi_i(\xi_i)|^{p_i} \left(b_i - \rho(\xi_i) \right) \diamond_\alpha \xi_i \right)^{\frac{1}{p_i}}. \tag{3.1}$$

Proof. Step 1: Representation of $|\psi_i(\xi_i)|$

From the diamond-alpha fundamental theorem (analog of Lemma 2.4)

$$|\psi_i(\xi_i)| = \int_{a_i}^{\xi_i} |\diamond_\propto \psi_i(t_i)| \diamond_\propto t_i \Rightarrow |\psi_i(\xi_i)| \cdot w_i(\xi_i) = w_i(\xi_i) \int_{a_i}^{\xi_i} |\diamond_\propto \psi_i(t_i)| \diamond_\propto t_i \qquad (3.2)$$

Step 2: Reverse Hölder's Inequality,Apply the reverse Hölder inequality for each term

$$w_i(\xi_i) \int_{a_i}^{\xi_i} |\diamond_\propto \psi_i(t_i)| \diamond_\propto t_i \geq \left(\int_{a_i}^{\xi_i} w_i(\xi_i)^{p_i} |\diamond_\propto \psi_i(t_i)|^{p_i} \diamond_\propto t_i \right)^{\frac{1}{p_i}} \left(\int_{a_i}^{\xi_i} 1 \diamond_\propto t_i \right)^{\frac{1}{q_i}} \qquad (3.3)$$

So we get

$$w_i(\xi_i)|\psi_i(\xi_i)| \geq (\xi_i - a_i)^{\frac{1}{q_i}} \left(\int_{a_i}^{\xi_i} w_i(\xi_i)^{p_i} |\diamond_\propto \psi_i(t_i)|^{p_i} \diamond_\propto t_i \right)^{\frac{1}{p_i}} \qquad (3.4)$$

Multiplying over all $\prod_{i=1}^{n}$

$$\prod_{i=1}^{n} w_i(\xi_i)|\psi_i(\xi_i)| \geq \prod_{i=1}^{n} (\xi_i - a_i)^{\frac{1}{q_i}} \left(\int_{a_i}^{\xi_i} w_i(\xi_i)^{p_i} |\diamond_\propto \psi_i(t_i)^{p_i} \diamond_\propto t_i \right)^{\frac{1}{p_i}} \qquad (3.5)$$

Step 3: Apply Mean Inequality (Lemma 2.6 modified)

$$\prod_{i=1}^{n} (\xi_i - a_i)^{1/q_i} \geq \frac{\left(\sum_{i=1}^{n} \frac{\xi_i - a_i}{q_i} \right)^{\sum_{i=1}^{n} \frac{1}{q_i}}}{\left(n - \sum_{i=1}^{n} \frac{1}{p_i} \right)^{n - \sum_{i=1}^{n} \frac{1}{p_i}}} \qquad (3.6)$$

Substituting in the previous result

$$\prod_{i=1}^{n} |\psi_i(\xi_i)| \geq \frac{\left(\sum_{i=1}^{n} \frac{\xi_i - a_i}{q_i} \right)^{\sum_{i=1}^{n} \frac{1}{q_i}}}{\left(n - \sum_{i=1}^{n} \frac{1}{p_i} \right)^{n - \sum_{i=1}^{n} \frac{1}{p_i}}} \prod_{i=1}^{n} \left(\int_{a_i}^{\xi_i} |\diamond_\propto \psi_i(t_i)|^{p_i} \diamond_\propto t_i \right)^{\frac{1}{p_i}} \qquad (3.7)$$

Step 4: Multiply Both Sides and Integrate

Multiply both sides by the normalizing factor and integrate over $\xi_i \in [a_i, b_i]$

$$\int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} \frac{\left(n - \sum_{i=1}^{n} \frac{1}{p_i} \right)^{n - \sum_{i=1}^{n} \frac{1}{p_i}}}{\left(\sum_{i=1}^{n} \frac{\xi_i - a_i}{q_i} \right)^{\sum_{i=1}^{n} \frac{1}{q_i}}} \prod_{i=1}^{n} (b_i - a_i)^{1/q_i} w_i(\xi_i)|\psi_i(\xi_i)| \diamond_\propto \xi_1 \dots \diamond_\propto \xi_n$$

$$\geq \prod_{i=1}^{n} \int_{a_i}^{b_i} \left(\int_{a_i}^{\xi_i} w_i(\xi_i)^{p_i} |\diamond_\propto \psi_i(t_i)|^{p_i} \diamond_\propto t_i \right)^{\frac{1}{p_i}} \diamond_\propto \xi_i. \qquad (3.8)$$

Step 5: Apply Reverse Minkowski and Integration by Parts (for $\diamond_\propto$)

Using an integration-by-parts type result adapted for $\diamond_\propto$,

$$\int_{a_i}^{b_i} \left(\int_{a_i}^{\xi_i} w_i(\xi_i) \,|\diamond_\propto \psi_i(t_i)|^{p_i} \diamond_\propto t_i \right)^{\frac{1}{p_i}} \diamond_\propto \xi_i$$

$$\geq \left(\int_{a_i}^{b_i} w_i(\xi_i)|\diamond_\propto \psi_i(\xi_i)|^{p_i}\big(b_i - \rho(\xi_i)\big) \diamond_\propto \xi_i \right)^{\frac{1}{p_i}} \qquad (3.9)$$

Corollary 3.1 Let $\mathbb{T} = \mathbb{R}, \rho(\xi_i) = \xi_i, a_i = 0, 0 < p_i < 1,$ and $q_i < 0$ such that $\frac{1}{p_i} + \frac{1}{q_i} = 1.$ Let either ψ_i be a positive, increasing satisfying $\psi_i(0) = 0$;moreover , for each $i = 1,2,\dots,n,$ the weight function $\rho(\xi_i)$ is positive and continuous, then we have

$$\int_0^{b_n} \cdots \int_0^{b_1} \frac{K \prod_{i=1}^n w_i(\xi_i)\big|\psi_i^{\diamond^\alpha}(\xi_i)\big|}{\left(\sum_{i=1}^n \frac{\xi_i}{q_i}\right)^{\sum_{i=1}^n \frac{1}{q_i}}} d_{\diamond^\alpha}\xi_1 \cdots d_{\diamond^\alpha}\xi_n$$

$$\geq \prod_{i=1}^n \left(\int_0^{b_i} w_i(\xi_i)\big|\psi_i^{\diamond^\alpha}(\xi_i)\big|^{p_i}[b_i - \xi_i]^{p_i} d_{\diamond^\alpha}\xi_i \right)^{\frac{1}{p_i}}$$

where ,

$$K = \frac{\left(n - \sum_{i=1}^n \frac{1}{p_i}\right)^{n - \sum_{i=1}^n \frac{1}{p_i}}}{\prod_{i=1}^n b_i^{\frac{1}{q_i}}}$$

is the best possible constant.

4.   CONCLUSIONS

In this paper, we established new reversed weighted dynamic inequalities of Hilbert type within the framework of diamond-alpha calculus on time scales. By employing essential analytical tools such as the reverse Hölder inequality, the reverse mean inequality, and integration by parts, we derived a unified inequality that simultaneously extends both continuous and discrete Hilbert-type inequalities. One of the main contributions of this study is that the obtained results generalize several known inequalities in the literature. In particular, by choosing the time scale T = R and T = N, the main results reduce to the classical reversed continuous and discrete Hilbert-type inequalities, respectively. Moreover, the introduction of appropriate weight functions provides additional flexibility and allows the results to be applied to a wider class of dynamic problems. The techniques developed in this paper may be extended to other types of dynamic operators and multidimensional inequalities on time scales.

Beyond their theoretical interest, these inequalities are significant for their potential applications in the stability analysis of dynamic equations and in determining the bounds for solutions in hybrid systems that exhibit both continuous and discrete

behaviors4. The introduction of appropriate weight functions w_i provides additional flexibility, allowing these results to be applied to a wider class of dynamic problems.

As future work, the techniques developed in this paper may be extended to other types of dynamic operators and multidimensional inequalities on time scales. These results can be further generalized to fractional-type inequalities or investigated within the framework of quantum calculus (q-calculus) on time scales.

$$\int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} \frac{\left(n - \sum_{i=1}^n \frac{1}{p_i}\right)^{n - \sum_{i=1}^n \frac{1}{n_i}}}{\left(\sum_{i=1}^n \frac{\xi_i - a_i}{q_i}\right)^{\sum_{i=1}^n \frac{1}{m_i}}} \prod_{i=1}^n (b_i - a_i)^{1/q_i} w_i(\xi_i) |\psi_i(\xi_i)| \diamond_\propto \xi_1 \dots \diamond_\propto \xi_n$$

$$\geq \prod_{i=1}^n \left(\int_{a_i}^{b_i} w_i(\xi_i) \ |\diamond_\propto \psi_i(\xi_i)^{p_i} \big(b_i - \rho(\xi_i)\big) \diamond_\propto \xi_i \right)^{1/p_i}$$

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