*Journal of Tr. Math. Sci., Vol. 3, No. 3, 2026, pp. 76-85*
*E-ISSN: 3062-3138*

Research Article

# Diamond-$\alpha$ Dynamic Type Inequalities

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**Abstract**
In this work, we establish diamond-$\alpha$ inequalities of dynamic type by applying the reversed Hölder inequality and the chain rule in diamond-$\alpha$ calculus on time scales. As particular cases of our results (when $\mathbb{T} = \mathbb{N}$ and $\mathbb{T} = \mathbb{R}$), we obtain the reversed forms of discrete and continuous inequalities.

**Keywords:** Hilbert-type inequalities, Diamond-$\alpha$ calculus, Hölder's inequality.

**MSC:** 26E70, 34N05, 30H10

**Received:** 25/03/2026 **Accepted:** 22/04/2026

## 1 INTRODUCTION

David Hilbert proved Hilbert's double-series inequality without exact determination of the constant in his lectures (see [1]) and proved that if $\{c_m\}_{m=1}^{\infty}$ and $\{d_n\}_{n=1}^{\infty}$ are two real sequences such that $0 < \sum_{m=1}^{\infty} c_m^2 < \infty$ and $0 < \sum_{n=1}^{\infty} d_n^2 < \infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{c_m d_n}{m+n} \le \pi\left(\sum_{m=1}^{\infty} c_m^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty} d_n^2\right)^{\frac{1}{2}} \tag{1.1}$$

In 1911, Schur [2] discovered the integral analogue of (1.1), which became known as the Hilbert integral inequality.

$$\int_0^{\infty}\int_0^{\infty}\frac{g(x)h(y)}{x+y}dxdy \le \pi\left(\int_0^{\infty} g^2(x)dx\right)^{\frac{1}{2}}\left(\int_0^{\infty} h^2(y)dy\right)^{\frac{1}{2}} \tag{1.2}$$

for real functions $g$ and $h$ such that $0 < \int_0^\infty g^2(x)dx < \infty$ and $0 < \int_0^\infty h^2(y)dy < \infty$.

The constant $\pi$ in (1.1) and (1.2) is the best possible constant factor. In 1925, by introducing a pair of conjugate exponents $(p,q)$ $(p,q > 1$ with $1/p + 1/q = 1)$. Hardy [3] gave an extension of (1.1) as follows. If $p,q > 1$ and $c_m, d_n \geq 0$ are such that $0 < \sum_{m=1}^\infty c_m^p < \infty$ and $0 < \sum_{n=1}^\infty d_n^q < \infty$ then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{c_m d_n}{m+n} \leq \frac{\pi}{\sin\frac{\pi}{p}}\left(\sum_{m=1}^{\infty}c_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}d_n^q\right)^{\frac{1}{q}} \tag{1.3}$$

Hardy and Reisz [1] proved the equivalent integral analogue of (1.3)).

$$\int_0^\infty\int_0^\infty \frac{g(x)h(y)}{x+y}dxdy \leq \frac{\pi}{\sin\frac{\pi}{p}}\left(\int_0^\infty g^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty h^q(y)dy\right)^{\frac{1}{q}} \tag{1.4}$$

for nonnegative functions $g$ and $h$ such that $0 < \int_0^\infty g^p(x)dx < \infty$ and $0 < \int_0^\infty h^q(y)dy < \infty$. The constant factor $\pi/\sin(\pi/p)$ in (1.3) and (1.4) is optimal. In 1998, Pachpatte [9] gave a new inequality close to that of Hilbert: Let $c(s) : \{0,1,2,\ldots,p\} \subset \mathbb{N} \to \mathbb{R}$ and $d(\vartheta) : \{0,1,2,\ldots,q\} \subset \mathbb{N} \to \mathbb{R}$ with $c(0) = d(0) = 0$.

Then

$$\sum_{s=1}^{p}\sum_{\vartheta=1}^{q}\frac{|c_s||d_\vartheta|}{s+\vartheta} \leq C(p,q)\left(\sum_{s=1}^{p}(p-s+1)|\nabla c_s|^2\right)^{\frac{1}{2}} \times \left(\sum_{\vartheta=1}^{q}(q-\vartheta+1)|\nabla d_\vartheta|^2\right)^{\frac{1}{2}} \tag{1.5}$$

where $\nabla c_s = c_s - c_{s-1}$ and $c(p,q) = \frac{1}{2}\sqrt{pq}$.

In 2002, Kim et al. [4] proved that if $\lambda, \mu > 1$ and $c(s) : \{0,1,2,\ldots,p\} \subset \mathbb{N} \to \mathbb{R}$ and $d(\vartheta) : \{0,1,2,\ldots,q\} \subset \mathbb{N} \to \mathbb{R}$ with $c(0) = d(0) = 0$, then

$$\begin{aligned}\sum_{s=1}^{p}\sum_{\vartheta=1}^{q}\frac{|c_s||d_\vartheta|}{\mu s^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}} + \lambda\vartheta^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}} &\leq D^*(\lambda,\mu,p,q)\left(\sum_{s=1}^{p}(p-s+1)|\nabla c_s|^\lambda\right)^{\frac{1}{\lambda}} \\ &\times\left(\sum_{\vartheta=1}^{q}(q-\vartheta+1)|\nabla d_\vartheta|^\mu\right)^{\frac{1}{\mu}}\end{aligned} \tag{1.6}$$

where $\nabla c_s = c_s - c_{s-1}$ and $D^*(\lambda,\mu,p,q) = \frac{1}{\lambda+\mu}p^{\frac{\lambda-1}{\lambda}}q^{\frac{\mu-1}{\mu}}$.

Also, Kim et al. [4] proved the continuous analogue of (1.6)

Let $\lambda, \mu > 1$, and let $g$ and $h$ be real continuous functions on the intervals $(0,x)$ and $(0,y)$, respectively, with $g(0) = h(0) = 0$. Then

$$\begin{aligned}&\int_0^x\int_0^y \frac{|g(s)||h(t)|}{\mu s^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}} + \lambda\vartheta^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}}dsdt \\ &\leq M(\lambda,\mu,x,y)\left(\int_0^x (x-s)|g'(s)|^\lambda ds\right)^{\frac{1}{\lambda}}\left(\int_0^y (y-t)|h'(t)|^\mu dt\right)^{\frac{1}{\mu}}\end{aligned} \tag{1.7}$$

for $x, y \in (0,\infty)$, where $n(\lambda,\mu,x,y) = \frac{1}{\lambda+\mu} x^{\frac{\lambda-1}{\lambda}} y^{\frac{\mu-1}{\mu}}$.

In 2011, Chang-Jian et al. [5] generalized (1.5) as follows. Let $p_i > 1$ and $1/p_i + 1/p_i^* = 1$, and let $c_i(s_i)$ be real sequences defined for $s_i = 0, 1, 2, \ldots, m_i$ such that $c_i(0) = 0, i = 1, 2, \ldots, n$. Define the operator $\nabla$ by $\nabla c_i(s_i) = c_i(s_i) - c_i(s_i - 1)$ for any function $c_i(s_i), i = 1, 2, \ldots, n$. Then

$$\begin{aligned} &\sum_{s_1=1}^{n_1} \cdots \sum_{s_n=1}^{n_n} \frac{\left(n - \sum_{i=1}^n \frac{1}{p_i}\right)^{\left(n - \sum_{i=1}^n \frac{1}{p_i^*}\right)}}{\prod_{i=1}^n m_i^{\frac{1}{p_i^*}}} \cdot \frac{\prod_{i=1}^n |a_i(s_i)|}{\left(\sum_{i=1}^n s_i/p_i^*\right)^{\sum_{i=1}^n \frac{1}{p_i^*}}} \\ &\leq \prod_{i=1}^n \left( \sum_{s_i=1}^{m_i} (m_i - s_i + 1) |\nabla c_i(s_i)|^{p_i} \right)^{\frac{1}{p_i}} \end{aligned} \tag{1.8}$$

Also, the authors of [5] proved that if $t_i \geq 1$ and $p_i > 1$ are constants, $1/p_i + 1/p_i^* = 1$, and $g_i(s_i)$ are real-valued differentiable functions on $[0, x_i)$, where $x_i \in (0,\infty)$, such that $g_i(0) = 0$ for $i = 1, 2, \ldots, n$, then

$$\begin{aligned} &\int_0^{x_1} \cdots \int_0^{x_n} \frac{\left(n - \sum_{i=1}^n \frac{1}{p_i}\right)^{\left(n - \sum_{i=1}^n \frac{1}{p_i^*}\right)}}{\prod_{i=1}^n t_i x_i^{\frac{1}{p_i^*}}} \frac{\prod_{i=1}^n |g_i^{t_i}(s_i)|}{\left(\sum_{i=1}^n s_i/p_i^*\right)^{\sum_{i=1}^n 1/p_i^*}} ds_n \ldots ds_1 \\ &\leq \prod_{i=1}^n \left( \int_0^{x_i} \frac{(x_i - s_i)}{g_i^{t_i-1}(s_i)} \cdot g_i'(s_i) ds_i \right)^{\frac{1}{p_i}} \end{aligned} \tag{1.9}$$

In recent years, a unified framework known as time scale theory has been developed to bridge the gap between continuous and discrete analysis. A time scale, denoted by $\mathbb{T}$, is defined as an arbitrary nonempty closed subset of the set of real numbers $\mathbb{R}$. This theory provides a powerful mathematical setting in which continuous and discrete cases can be studied simultaneously. The results presented in this paper encompass both the classical continuous and discrete inequalities as particular cases corresponding to $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{N}$, respectively. Furthermore, these inequalities can be naturally extended to various other time scales such as $\mathbb{T} = h^{\mathbb{N}}, h > 0$, and $\mathbb{T} = q^{\mathbb{N}}$ for $q > 1$.

Over the past decades, a considerable amount of research has been devoted to the study of dynamic inequalities on time scales, significantly enriching the theoretical background of the field. The main objective of the present work is to establish sufficient conditions for obtaining new reversed forms of inequalities (1.8) and (1.9) on time scales by developing a class of novel Hilbert-type inequalities within the framework of nabla calculus. The proofs of the main results are carried out by employing integration by parts, the reverse Hölder inequality on time scales, and the reverse mean inequality.

A wide range of studies has focused on extending classical inequalities within the framework of time scales. In this direction, various generalizations of Hardy-type and Hilbert-type inequalities have been established by considering delta, nabla, and diamond-$\alpha$ operators. These approaches provide a unified treatment of continuous and discrete cases and enable the derivation of more general forms of inequalities.

In particular, Saied [13] investigated reversed dynamic inequalities of Hilbert-type within the nabla calculus setting. Furthermore, Akın et al. [14–16] introduced several new results concerning fractional and multidimensional dynamic inequalities on time scales. These contributions highlight the significance of the diamond-$\alpha$ operator in obtaining unified and flexible formulations of inequalities.

However, despite these developments, certain aspects of reversed inequalities within the diamond-$\alpha$ framework remain less explored. This observation motivates the present study, in which new reversed dynamic inequalities of Hilbert-type are established under more general conditions.

# 2 PRELIMINARIES

The concept of time scales was first introduced in Stephen Hilger's [18] doctoral dissertation in 1988. Later, in 2001, Bohner and Peterson [6] developed this concept further and defined the time scale $\mathbb{T}$ as an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. Also, they defined the backward jump operator as $\rho(\tau) := \sup\{s \in \mathbb{T} : s < \tau\}$. For any function $f : \mathbb{T} \to \mathbb{R}$, $g^{\rho}(\tau)$ denotes $g(\rho(\tau))$. We define the time scale interval $[c, d]_{\mathbb{T}}$ by $[c, d]_{\mathbb{T}} = [c, d] \cap \mathbb{T}$. For more details, see [14–16].

**Definition 2.1** [7] A function $\lambda : \mathbb{T} \to \mathbb{R}$ is left-dense continuous or $ld$-continuous if it is continuous at left-dense points in $\mathbb{T}$ and its right-sided limits exist at right-dense points in $\mathbb{T}$. The space of $ld$-continuous functions is denoted by $C(\mathbb{T}, \mathbb{R})$.

**Definition 2.2** [8] Let $\mathbb{T}$ be a time scale, $g : \mathbb{T} \to \mathbb{R}$ a function, and $k \in \mathbb{T}^k_k$. In this case, for $\alpha \in [0, 1]$, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $s$ in a neighborhood $U$ of the point $k$, with $\mu_{ts} = \sigma(k) - s$ and $\nu_{ks} = \rho(k) - s$, the inequality

$$|\alpha[g^{\sigma}(k) - g(s)]\nu_{ks} + (1 - \alpha)[g^{\rho}(k) - g(s)]\mu_{ks} - g^{\diamond_\alpha}(k)\mu_{ks}\nu_{ks}| < \varepsilon|\mu_{ks}\nu_{ks}|$$

holds. If this inequality is satisfied, then the expression $g^{\diamond_\alpha}(k)$ is called the diamond-$\alpha$ derivative of the function $g$ on $\mathbb{T}^k_k$.

**Lemma 2.1** [8] Let $g : \mathbb{R} \to \mathbb{R}$ be a differentiable function and let $h : \mathbb{T} \to \mathbb{R}$ be a function that is diamond-$\alpha$ differentiable at $k \in \mathbb{T}^k_k$. Then the composition $g \circ h$ is diamond-$\alpha$ differentiable at $k$, and its diamond-$\alpha$ derivative is given by:

$$(g \circ h)^{\diamond_\alpha}(k) = g'(h(k)) \cdot h^{\diamond_\alpha}(k)$$

where $g$ is the usual derivative of $g$ and $h^{\diamond_\alpha}(k)$ is the diamond-$\alpha$ derivative of $h$ at point $k$.

**Definition 2.3** [13] Let $a, t \in \mathbb{T}$ and $g : \mathbb{T} \to \mathbb{R}$ be a function. In this case, for $\alpha \in [0, 1]$, the $\diamond_\alpha$ integral is defined as

$$\int_a^t g(\tau) \diamond_\alpha \tau = \alpha \int_a^t g(\tau)\Delta\tau + (1 - \alpha) \int_a^t g(\tau)\nabla\tau$$

The $\diamond_\alpha$ integral is a linear combination of the $\Delta$ (delta) and $\nabla$ (nabla) integrals.

In general, for $t \in \mathbb{T}$, the equality

$$\left(\int_a^t g(\tau) \diamond_\alpha \tau\right)^{\diamond_\alpha} = g(t)$$

does not hold.

**Theorem 2.1** [9] Let $a, b, t \in \mathbb{T}$ and $c \in \mathbb{R}$, $g, h : \mathbb{T} \to \mathbb{R}$. In this case,

(i)
$$\int_a^t [g(\tau) + h(\tau)] \diamond_\alpha \tau = \int_a^t g(\tau) \diamond_\alpha \tau + \int_a^t h(\tau) \diamond_\alpha \tau$$

(ii)
$$\int_a^t cg(\tau) \diamond_\alpha \tau = c \int_a^t g(\tau) \diamond_\alpha \tau$$

(iii)
$$\int_a^t g(\tau) \diamond_\alpha \tau = -\int_t^a g(\tau) \diamond_\alpha \tau$$

(iv)
$$\int_a^t g(\tau) \diamond_\alpha \tau = \int_a^b g(\tau) \diamond_\alpha \tau + \int_b^t g(\tau) \diamond_\alpha \tau$$

(v)
$$\int_a^a g(\tau) \diamond_\alpha \tau = 0$$

**Lemma 2.2** [10] (Integration by parts using the diamond-$\alpha$ integral) Let $a, b \in \mathbb{T}$, and let $g, h : \mathbb{T} \to \mathbb{R}$ be $ld$-continuous functions. Then, the integration by parts formula concerning the diamond-$\alpha$ integral is given by:

$$\int_a^b g(t) h^{\diamond_\alpha}(t) \diamond_\alpha t = [g(t)h(t)]_a^b - \int_a^b g^{\diamond_\alpha}(t) h(t) \diamond_\alpha t$$

provided the involved integrals and derivatives exist.
**Lemma 2.3** [11]Let $g, h : \mathbb{T} \to \mathbb{F}$ be positive functions that are diamond-$\alpha$ integrable on the interval $[a, b] \subset \mathbb{T}$, where $\mathbb{T}$ is a time scale and $\alpha \in [0, 1]$. If $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the reverse Hölder inequality in the context of diamond-$\alpha$ integrals is given by:

$$\int_a^b g(t)h(t) \diamond_\alpha t \geq \left( \int_a^b g^p(t) \diamond_\alpha t \right)^{\frac{1}{p}} \left( \int_a^b h(t) \diamond_\alpha t \right)^{\frac{1}{q}}$$

provided that the right-hand side integrals exist and are finite.

**Lemma 2.4** [11]Let $c_i, d_i \in \mathbb{T}$, and let either $\varphi_i \in C([c_i, d_i]_{\mathbb{T}}, (-\infty, 0])$ be nonincreasing functions or $\varphi_i \in C([c_i, d_i]_{\mathbb{T}}, [0, \infty))$ be nondecreasing functions with $\varphi_i(c_i) = 0$, $i = 1, 2, \ldots, n$. Then
$$\int_{c_i}^{\xi_i} |\varphi_i^{\diamond_\alpha}(t_i)| \diamond_\alpha t_i = |\varphi_i(\xi_i)|, \xi_i \in [c_i, d_i]_{\mathbb{T}}$$

**Lemma 2.5** (Mean inequality [1]) If $\alpha_i, \theta_i > 0$ for $i = 1, 2, \ldots, n$, then

$$\prod_{i=1}^n \alpha_i^{\theta_i} \leq \frac{\left(\sum_{i=1}^n \alpha_i \theta_i\right)^{\left(\sum_{i=1}^n \theta_i\right)}}{\left(\sum_{i=1}^n \theta_i\right)^{\left(\sum_{i=1}^n \theta_i\right)}}.$$

**Lemma 2.6** [1] Let $q_i < 0$ with $1/p_i + 1/q_i = 1$, and let $s_i > 0, i = 1, 2, \ldots, n$. Then

$$\prod_{i=1}^{n} s_i^{1/q_i} \geq \frac{\left(\sum_{i=1}^{n} \frac{s_i}{q_i}\right)^{\left(\sum_{i=1}^{n} \frac{1}{q_i}\right)}}{\left(n - \sum_{i=1}^{n} \frac{1}{p_i}\right)^{\left(n - \sum_{i=1}^{n} \frac{1}{p_i}\right)}}$$

# 3 RESULTS

**Theorem 3.1** Let $c_i, d_i \in \mathbb{T}, h_i \geq 1, q_i < 0, p_i = \frac{q_i}{q_i - 1}$, and let $\varphi_i \in C([c_i, d_i]_{\mathbb{T}}, [0, \infty))$ be decreasing functions such that $\varphi_i(d_i) = 0$, for all $i = 1, 2, \ldots, n$. Let also $w_i : [c_i, d_i]_{\mathbb{T}} \to \mathbb{R}^+$ be a positive weight function for all $i = 1, 2, \ldots, n$. Then

$$\begin{aligned} &\int_{c_n}^{d_n} \cdots \int_{c_1}^{d_1} \frac{\left(n - \sum_{i=1}^{n} \frac{1}{p_i}\right)^{\left(n - \sum_{i=1}^{n} \frac{1}{p_i}\right)} \prod_{i=1}^{n} \varphi_i^{h_i}(\xi_i) w_i(\xi_i)}{\left(\sum_{i=1}^{n} \frac{b_i - \xi_i}{q_i}\right)^{\left(\sum_{i=1}^{n} \frac{1}{q_i}\right)}} \diamond_\alpha \xi_1 \cdots \diamond_\alpha \xi_n \\ &\geq \prod_{i=1}^{n} h_i (d_i - c_i)^{\frac{1}{q_i}} \left( \int_{c_i}^{d_i} (\rho(\xi_i) - c_i) [\varphi_i(\xi_i)^{h_i - 1} (- \diamond_\alpha \varphi_i(\xi_i)) w_i(\xi_i)]^{p_i} \diamond_\alpha \xi_i \right)^{\frac{1}{p_i}} \end{aligned} \tag{3.1}$$

**Proof:**
**Step 1:** Chain Rule for Diamond-$\alpha$ Derivative
Using the diamond-$\alpha$ chain rule for $\varphi_i^{h_i}(k_i)$, with $h_i \geq 1$ :

$$\diamond_\alpha [-\varphi_i^{h_i}(k_i)] = h_i \cdot \varphi_i^{h_i - 1}(\zeta_i) \cdot (- \diamond_\alpha \varphi_i(k_i)), \zeta_i \in [\rho(k_i), k_i]$$

Since $\varphi_i$ is decreasing and $\zeta_i \leq k_i$, we have:

$$\diamond_\alpha [-\varphi_i^{h_i}(k_i)] \geq h_i \cdot \varphi_i^{h_i - 1}(k_i) \cdot (- \diamond_\alpha \varphi_i(k_i))$$

Integrating both sides from $\xi_i$ to $d_i$, and using $\varphi_i(d_i) = 0$:

$$h_i \int_{\xi_i}^{d_i} \varphi_i^{h_i - 1}(k_i)(- \diamond_\alpha \varphi_i(k_i)) w_i(k_i) \diamond_\alpha k_i \leq \varphi_i^{h_i}(\xi_i) w_i(\xi_i)$$

Hence:

$$\prod_{i=1}^{n} \varphi_i^{h_i}(\xi_i) w_i(\xi_i) \geq \prod_{i=1}^{n} h_i \int_{\xi_i}^{d_i} \varphi_i^{h_i - 1}(k_i)(- \diamond_\alpha \varphi_i(k_i)) w_i(k_i) \diamond_\alpha k_i$$

**Step 2:** Reverse Hölder's Inequality (applied to the integrals)
Applying reverse Hölder's inequality:

$$\begin{aligned} &\int_{\xi_i}^{d_i} \varphi_i^{h_i - 1}(k_i)(- \diamond_\alpha \varphi_i(k_i)) w_i(k_i) \diamond_\alpha k_i \\ &\geq (d_i - \xi_i)^{\frac{1}{q_i}} \left( \int_{\xi_i}^{d_i} [\varphi_i^{h_i - 1}(k_i)(- \diamond_\alpha \varphi_i(k_i)) w_i(k_i)]^{p_i} \diamond_\alpha k_i \right)^{\frac{1}{p_i}} \end{aligned}$$

Multiplying over $i$ :

$$\prod_{i=1}^{n}\varphi_i^{h_i}(\xi_i)\geq\prod_{i=1}^{n}h_i(d_i-\xi_i)^{\frac{1}{q_i}}\left(\int_{\xi_i}^{d_i}[\varphi_i^{h_i-1}(k_i)(-\diamond_\alpha\varphi_i(k_i))]^{p_i}\diamond_\alpha k_i\right)^{\frac{1}{p_i}}$$

**Step 3:** Mean Inequality (reverse of AM-GM)
By the generalised mean inequality (Lemma 2.1):

$$\prod_{i=1}^{n}(d_i-\xi_i)^{\frac{1}{q_i}}\geq\frac{\left(\sum_{i=1}^{n}\frac{d_i-\xi_i}{q_i}\right)^{\left(\sum_{i=1}^{n}\frac{1}{q_i}\right)}}{\left(n-\sum_{i=1}^{n}\frac{1}{p_i}\right)^{\left(n-\sum_{i=1}^{n}\frac{1}{p_i}\right)}}$$

Substitute into the previous inequality:

$$\prod_{i=1}^{n}\varphi_i^{h_i}(\xi_i)\geq\frac{\left(\sum_{i=1}^{n}\frac{d_i-\xi_i}{q_i}\right)^{\left(\sum_{i=1}^{n}\frac{1}{q_i}\right)}}{\left(n-\sum_{i=1}^{n}\frac{1}{p_i}\right)^{\left(n-\sum_{i=1}^{n}\frac{1}{p_i}\right)}}\prod_{i=1}^{n}h_i\left(\int_{\xi_i}^{d_i}[\varphi_i^{h_i-1}(k_i)(-\diamond_\alpha\varphi_i(k_i))]^{p_i}\diamond_\alpha k_i\right)^{\frac{1}{p_i}}$$

**Step 4:** Multiply both sides and integrate over $\xi_i\in[c_i,d_i]$

$$\begin{aligned}&\int_{c_n}^{d_n}\cdots\int_{c_1}^{d_1}\frac{\left(n-\sum_{i=1}^{n}\frac{1}{p_i}\right)^{\left(n-\sum_{i=1}^{n}\frac{1}{p_i}\right)}}{\left(\sum_{i=1}^{n}\frac{d_i-\xi_i}{q_i}\right)^{\left(\sum_{i=1}^{n}\frac{1}{q_i}\right)}}\prod_{i=1}^{n}\varphi_i^{h_i}(\xi_i)w_i(\xi_i)\diamond_\alpha\xi_1\cdots\diamond_\alpha\xi_n\\&\geq\prod_{i=1}^{n}h_i\int_{c_i}^{d_i}\left(\int_{\xi_i}^{d_i}[\varphi_i^{h_i-1}(k_i)(-\diamond_\alpha\varphi_i(k_i))w_i(k_i)]^{p_i}\diamond_\alpha k_i\right)^{\frac{1}{p_i}}\diamond_\alpha\xi_i\end{aligned}$$

**Step 5:** Integration by Parts for Diamond-$\alpha$
Using the $\diamond_\alpha$ integration-by-parts identity:

$$\begin{aligned}&\int_{c_i}^{d_i}\left(\int_{\xi_i}^{d_i}[\varphi_i^{h_i-1}(k_i)(-\diamond_\alpha\varphi_i(k_i))w_i(k_i)]^{p_i}\diamond_\alpha k_i\right)w_i(\xi_i)\diamond_\alpha\xi_i\\&=\int_{c_i}^{d_i}(\rho(\xi_i)-c_i)[\varphi_i^{h_i-1}(\xi_i)(-\diamond_\alpha\varphi_i(\xi_i))]^{p_i}w_i(\xi_i)\diamond_\alpha\xi_i\end{aligned}$$

Final Inequality (as claimed)

$$\begin{aligned}&\int_{c_n}^{d_n}\cdots\int_{c_1}^{d_1}\frac{\left(n-\sum_{i=1}^{n}\frac{1}{p_i}\right)^{\left(n-\sum_{i=1}^{n}\frac{1}{p_i}\right)}\prod_{i=1}^{n}\varphi_i^{h_i}(\xi_i)}{\left(\sum_{i=1}^{n}\frac{d_i-\xi_i}{q_i}\right)^{\left(\sum_{i=1}^{n}\frac{1}{q_i}\right)}}\prod_{i=1}^{n}\varphi_i^{h_i}(\xi_i)w_i(\xi_i)\diamond_\alpha\xi_1\cdots\diamond_\alpha\xi_n\\&\geq\prod_{i=1}^{n}h_i(d_i-c_i)^{\frac{1}{q_i}}\left(\int_{c_i}^{d_i}(\rho(\xi_i)-c_i)[\varphi_i^{h_i-1}(\xi_i)(-\diamond_\alpha\varphi_i(\xi_i))w_i(\xi_i)]^{p_i}\diamond_\alpha\xi_i\right)^{\frac{1}{p_i}}.\end{aligned}$$

**Corollary 3.1** Let $\mathbb{T}=\mathbb{R}, c_i, d_i \in \mathbb{T}, h_i \geq 1, q_i<0, p_i=\frac{q_i}{q_i-1}$, and let $\varphi_i \in C([c_i, d_i],[0, \infty))$ be a decreasing function such that $\varphi_i(d_i)=0$, for $i=1,2, \ldots, n$. Then $\rho(\xi_i)=\xi_i$, and the following inequality holds:

$$\int_{c_n}^{d_n} \cdots \int_{c_1}^{d_1} \frac{\left(n-\sum_{i=1}^n \frac{1}{p_i}\right)^{\left(n-\sum_{i=1}^n \frac{1}{p_i}\right)}}{\left(\sum_{i=1}^n \frac{d_n-\xi_i}{q_i}\right)^{\left(\sum_{i=1}^n \frac{1}{q_i}\right)}} \prod_{i=1}^n \varphi_i^{h_i}(\xi_i) w_i(\xi_i) d \xi_1 \ldots d \xi_n$$
$$\geq \prod_{i=1}^n h_i(d_i-c_i)^{\frac{1}{q_i}}\left(\int_{c_i}^{d_i}(\xi_i-c_i)[\varphi_i^{h_i-1}(\xi_i) \cdot(-\diamond_\alpha \varphi_i(\xi_i)) w_i(\xi_i)]^{p_i} d \xi_i\right)^{\frac{1}{p_i}} .$$

Here, the delta-nabla structure degenerates into classical integration since $\mathbb{T}=\mathbb{R}$, but the derivative $\diamond_\alpha \varphi_i(\xi_i)$ remains defined as:

$$\diamond_\alpha \varphi_i(\xi_i)=\alpha \cdot \Delta \varphi_i(\xi_i)+(1-\alpha) \cdot \nabla \varphi_i(\xi_i).$$

**Corollary 3.2** Let $\mathbb{T}=\mathbb{N}_C, c_i, d_i \in \mathbb{T}, h_i \geq 1, q_i<0, p_i=\frac{q_i}{q_i-1}$, and let $\varphi_i$ be a nonnegative decreasing sequence such that $\varphi_i(d_i)=0$, for $i=1,2, \ldots, n$. Then $\rho(\xi_i)=\xi_i-1$, and the following inequality holds:

$$\sum_{\xi_n=c_n+1}^{d_n} \cdots \sum_{\xi_1=c_1+1}^{d_1} \frac{\left(n-\sum_{i=1}^n \frac{1}{p_i}\right)^{\left(n-\sum_{i=1}^n \frac{1}{p_i}\right)}}{\left(\sum_{i=1}^n \frac{d_i-\xi_i}{q_i}\right)^{\left(\sum_{i=1}^n \frac{1}{q_i}\right)}} \prod_{i=1}^n \varphi_i^{h_i}(\xi_i) w_i(\xi_i)$$
$$\geq \prod_{i=1}^n h_i(d_i-c_i)^{\frac{1}{q_i}}\left(\sum_{\xi_i=c_i+1}^{d_i}(\xi_i-c_i-1)[\varphi_i^{h_i-1}(\xi_i) \cdot(-\diamond_\alpha \varphi_i(\xi_i)) w_i(\xi_i)]^{p_i}\right)^{\frac{1}{p_i}} .$$

Here, the diamond-$\alpha$ difference is:

$$\diamond_\alpha \varphi_i(\xi_i)=\alpha \cdot(\varphi_i(\xi_i+1)-\varphi_i(\xi_i))+(1-\alpha) \cdot(\varphi_i(\xi_i)-\varphi_i(\xi_i-1)).$$

# 4 CONCLUSIONS

In this study, reverse Hilbert-type inequalities based on the diamond-$\alpha$ calculus for functions defined on time scales are investigated and generalized. New dynamic-type inequalities are established by employing generalized Hölder and mean-type inequalities. With the aid of expressions involving weight functions, particular results are derived for both continuous and discrete cases. Furthermore, limiting cases obtained by reducing the delta-nabla structure to the classical integration setting are also presented. The results obtained are expected to contribute to both the theoretical development of time scales calculus and its applications in various areas of applied mathematics.

# Author Contributions

For research articles with several authors, a short paragraph specifying their individual contributions must be provided.

## Acknowledgments

[17] The abstract of this study was presented at the ICOMAA 2025 International Conference, İstanbul, Türkiye)

## Conflicts of Interest

Declare conflicts of interest or state "The authors declare no conflict of interest."

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