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Research Article

Optimal Scaling in Double-Contact Regular Polygon Containment: Double-Contact Polygon Scaling

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Abstract
This study presents a detailed derivation of a trigonometric identity governing the optimal scaling of a regular m-gon inscribed within a regular n-gon underdouble-contact constraints. Building on prior work that established containment inequalities for nested polygons in the complex plane, we focus on the symmetric configuration where rotational and vertical translation components vanish (b = 0, d = 0). In this setting, we derive a closed-form expression for the scaling factor c by equating two distinct contact conditions involving edge-vertex interactions. The resulting identity incorporates cosine and cotangent terms and reveals how geometric symmetry leads to algebraic simplification. We also provide a long-form factorization and numerical examples to illustrate the identity's behavior across different polygon pairs. This work contributes to the broader theory of polygonal optimization and symbolic encoding in geometric configurations.

Keywords: circle packings and tangency; geometric inequalities and containment; plane geometry of polygons; complex analytic optimization
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1. INTRODUCTION

Geometry has long served as a bridge between scientific reasoning and artistic expression. From ancient civilizations such as Egypt and Mesopotamia to classical Greece, geometric principles were used to encode cosmic order and architectural harmony. Thinkers like Pythagoras and Euclid formalized foundational ideas through symmetry, proportion, and regular polygons. In the Islamic world, particularly during the Seljuk and Ottoman periods, geometry evolved into both a scientific tool and a spiritual medium. Intricate polygonal patterns in mosques and madrasas symbolized infinite repetition and divine unity [1, 8, 11].

In modern science and engineering, geometric modeling plays a central role in fields such as computer-aided design, materials science, and architectural optimization. Regular polygons, due to their inherent symmetry, are ideal candidates for nesting and packing problems. Numerous studies have explored how to inscribe one polygon within another to maximize area or minimize perimeter under geometric constraints [4, 5, 7, 12]. These problems have practical applications in solar energy systems [2, 3], cutting and packing algorithms [9, 10], and symbolic design

While traditional approaches rely heavily on coordinate geometry and trigonometric identities, such methods can become cumbersome when addressing rotational symmetry and complex transformations. Representing polygon vertices in the complex plane offers a more elegant and unified framework. Complex numbers allow scaling, rotation, and translation to be expressed algebraically, simplifying constraint formulation and enhancing analytical tractability.

Recent advances have introduced algorithmic and analytical techniques for inscribed polygon optimization. Notable contributions include dynamic programming for convex *k*-gons [14], maximum-area configurations in closed regions [16], and heuristic solutions for geometric enclosure problems [17]. Studies on rectangular and triangular optimization further highlight the computational depth of these problems [6, 12].

Building on this foundation, our previous work [13] introduced a complex-numberbased inequality for determining whether a regular *m*-gon is fully contained within a regular *n*-gon. In the present study, we extend that framework by analyzing double-contact configurations—cases where two distinct vertices of the inner polygon simultaneously touch two edges of the outer polygon. This leads to a new trigonometric identity for the scaling factor, derived under symmetry and alignment constraints.

2. Background and Prior Framework

This study builds upon the geometric and algebraic framework introduced in our earlier publication, Optimal Positioning of Regular Polygons under Area Constraints [13]. In that work, we investigated how a regular m-gon can be optimally inscribed within a regular n-gon to maximize its area, under strict containment conditions.

The polygons were modeled in the complex plane: the outer n-gon was centered at the origin with unit circumradius, and the inner m-gon was allowed to scale, rotate, and translate. Each vertex of the outer polygon was represented as $z_k = e^{2\pi ik/n}$ while the inner polygon was expressed as $w_\ell = \theta + \lambda e^{2\pi i\ell/m}$, where $\theta \in C$ is the center and $\lambda \in C$ encodes scale and orientation.

A key result was the derivation of a containment inequality ensuring that each vertex of the inner polygon lies on the correct side of every edge of the outer polygon.

This led to the real inequality:

$$a.\cos\left(2\pi\left(\frac{l}{m}-\frac{k+\frac{1}{2}}{n}\right)\right)+b.\sin\left(2\pi\left(\frac{l}{m}-\frac{k+\frac{1}{2}}{n}\right)\right)+c.\cos\left(2\pi\frac{k+\frac{1}{2}}{n}\right)+d.\sin\left(2\pi\frac{k+\frac{1}{2}}{n}\right)\leq\cos\left(\frac{\pi}{n}\right) \quad (1)$$

where a,b are the real and imaginary parts of λ, and c,d are the real and imaginary parts of θ.

This inequality formed the basis for optimizing the scale and placement of the inner polygon. Special cases such as coincident centers, pure scaling, and symmetric configura tions (a+c = 1) were explored. The expression |2ℓn−2km−m| was introduced as a key metric for identifying critical contact configurations.

$$a.\cos\left(2\pi\left(\frac{l}{m}-\frac{k+\frac{1}{2}}{n}\right)\right)+b.\sin\left(2\pi\left(\frac{l}{m}-\frac{k+\frac{1}{2}}{n}\right)\right)+c.\cos\left(2\pi\frac{k+\frac{1}{2}}{n}\right)+d.\sin\left(2\pi\frac{k+\frac{1}{2}}{n}\right)=\cos\left(\frac{\pi}{n}\right) \quad (2)$$

Equation (2) defines the precise configuration in which a vertex of the inner regular m-gon touches an edge of the outer regular n-gon. The parameters a and b represent the real and imaginary components of the inner polygon's scaling and rotation, while c and d denote its center coordinates. The left-hand side of the equation computes the projected position of the vertex relative to the edge midpoint, and the right-hand side cos π n corresponds to the angular threshold defined by the outer polygon's edge. When equality holds, the vertex lies exactly on the boundary, indicating a contact point. This condition is critical for identifying maximal configurations and bounding the scale factor under geometric constraints.

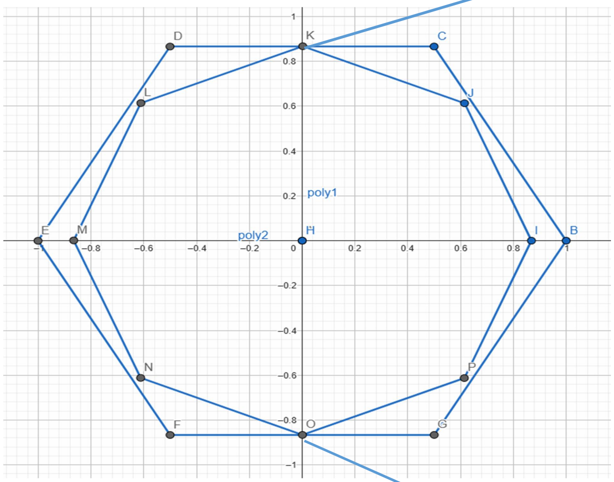
In the present work, we extend this framework by introducing double-contact constraints and analyzing the special case where the rotational and vertical translation components vanish, i.e., $b= 0$ and $d = 0$. Under this condition, the trigonometric inequality simplifies, allowing us to derive a new identity for the scaling factor based on symmetric edge-vertex alignment.

3. Main Result : ScalingIdentity under Double-Contact Constraints

Let n, m ≥ 3. The outer n-gon has unit circumradius; the inner m-gon has scale c > 0. For a touching between the ℓ-th vertex of the m-gon and the edge between the k-th and (k +1)-th vertices of the n-gon, the scale-at-contact

$$a(k,l)=\frac{\cos\left(\frac{\pi}{n}\right)-c.\cos\left(2\pi\frac{k+\frac{1}{2}}{n}\right)}{\cos\left(2\pi\left(\frac{l}{m}-\frac{k+\frac{1}{2}}{n}\right)\right)} \quad (3)$$

Let us analyze the case where n=6 and m=8.



Edge DC → $k_1 = \frac{3}{2}$

Edge FG → $k_1 = -\frac{3}{2}$

In a *double-contact* we have two distinct half-integers $k_1 + \frac{1}{2}$ and $k_2 + \frac{1}{2}$ in $(0, \frac{n}{2}]$ and indices $\ell_1, \ell_2 \in \{0, 1, \ldots, m - 1\}$ such that

$$a(k_1, \ell_1) = a(k_2, \ell_2) \tag{4}$$

Write k_1, k_2 for the half-indices (i.e. the formula already includes +12) and set, for later compactness,

$$\alpha_\pm := \frac{\pi(k_1 \pm k_2)}{n} \quad , \quad \beta_\pm := \frac{\pi(l_1 \pm l_2)}{m}$$

From (3)–(4):

$$\frac{\cos\left(\frac{\pi}{n}\right) - c\cos\left(2\pi\frac{k_1}{n}\right)}{\cos\left(2\pi\left(\frac{l_1}{m} - \frac{k_1}{n}\right)\right)} = \frac{\cos\left(\frac{\pi}{n}\right) - c\cos\left(2\pi\frac{k_2}{n}\right)}{\cos\left(2\pi\left(\frac{l_2}{m} - \frac{k_2}{n}\right)\right)} \Leftrightarrow$$

$$\left(\cos\left(\frac{\pi}{n}\right) - c.\cos\left(2\pi\frac{k_1}{n}\right)\right)\cos\left(2\pi\left(\frac{l_2}{m} - \frac{k_2}{n}\right)\right) = \left(\cos\left(\frac{\pi}{n}\right) - c\cos\left(2\pi\frac{k_2}{n}\right)\right)\cos\left(2\pi\left(\frac{l_1}{m} - \frac{k_1}{n}\right)\right) \tag{5}$$

Move all terms to opposite sides and group the cos(π/n) and c parts:

$$\cos\left(\frac{\pi}{n}\right)\left(\cos\left(2\pi\left(\frac{l_2}{m}-\frac{k_2}{n}\right)\right)-\cos\left(2\pi\left(\frac{l_1}{m}-\frac{k_1}{n}\right)\right)\right)=$$

$$=c\left(\cos\left(2\pi\frac{k_1}{n}\right)\cos\left(2\pi\left(\frac{l_2}{m}-\frac{k_2}{n}\right)\right)-c\cos\left(2\pi\frac{k_2}{n}\right)\cos\left(2\pi\left(\frac{l_1}{m}-\frac{k_1}{n}\right)\right)\right) \quad (6)$$

Use

$$\cos(u)-\cos(v)=-2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right),\ \cos(u)\cos(v)=\frac{1}{2}(\cos(u+v)+\cos(u-v))$$

For the left-hand side(LHS)of (6):

$$LHS=\cos\left(\frac{\pi}{n}\right)\left(-2\sin\left(\frac{U+V}{2}\right)\sin\left(\frac{U-V}{2}\right)\right) \quad (7)$$

With

$$U:=2\pi\left(\frac{l_2}{m}-\frac{k_2}{n}\right),\ V:=2\pi\left(\frac{l_1}{m}-\frac{k_1}{n}\right)$$

Compute the half-sums:

$$\frac{U+V}{2}=\pi\left(\frac{l_1+l_2}{m}-\frac{k_1+k_2}{n}\right)=\beta_+-\alpha_+$$

$$\frac{U-V}{2}=\pi\left(\frac{l_1-l_2}{m}-\frac{k_1-k_2}{n}\right)=\beta_--\alpha_-$$

Thus

$$LHS=-2\cos\left(\frac{\pi}{n}\right)\sin(\beta_+-\alpha_+)\sin(\beta_--\alpha_-)$$
(8)

For the right-hand side (RHS)of (6),expand each product:

$$\cos\left(\frac{2\pi k_1}{n}\right)\cos\left(2\pi\left(\frac{l_2}{m}-\frac{k_2}{n}\right)\right)=\frac{1}{2}\left(\cos\left(2\pi\left(\frac{l_2}{m}-\frac{k_2}{n}\right)+2\pi\frac{k_1}{n}\right)+\cos\left(2\pi\left(\frac{l_2}{m}-\frac{k_2}{n}\right)-2\pi\frac{k_1}{n}\right)\right)$$

$$\cos\left(\frac{2\pi k_2}{n}\right)\cos\left(2\pi\left(\frac{l_1}{m}-\frac{k_1}{n}\right)\right)=\frac{1}{2}\left(\cos\left(2\pi\left(\frac{l_1}{m}-\frac{k_1}{n}\right)+2\pi\frac{k_2}{n}\right)+\cos\left(2\pi\left(\frac{l_1}{m}-\frac{k_1}{n}\right)-2\pi\frac{k_2}{n}\right)\right)$$

Subtract the second from the first and group terms:

$$\frac{RHS}{c} = \frac{1}{2}\left(\cos\left(2\pi\frac{l_2}{m} - 2\pi\frac{k_2 - k_1}{n}\right) - \cos\left(2\pi\frac{l_1}{m} - 2\pi\frac{k_1 - k_2}{n}\right)\right) +$$
$$+\frac{1}{2}\left(\cos\left(2\pi\frac{l_2}{m} - 2\pi\frac{k_2+k_1}{n}\right) - \cos\left(2\pi\frac{l_1}{m} - 2\pi\frac{k_1+k_2}{n}\right)\right) \quad (9)$$

Each difference $\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$.

For the first bracketi in (9):

$$A_1 = 2\pi\left(\frac{l_2}{m} - \frac{k_2 - k_1}{n}\right), B_1 = 2\pi\left(\frac{l_1}{m} - \frac{k_1 - k_2}{n}\right)$$

$$\frac{A_1 + B_1}{2} = \pi\frac{l_1 - l_2}{n} = \beta_+, \frac{A_1 - B_1}{2} = \pi\left(\frac{l_2 - l_1}{m} - \frac{2(k_2 - k_1)}{n}\right) = -(\beta_- - 2\alpha_-)$$

Hence

$\cos A_1 - \cos B_1 = -2\sin\beta_+\sin(-(\beta_- - 2\alpha_-)) = 2\sin\beta_+\sin(\beta_- - 2\alpha_-)$.

Fort he second bracket ,n (9);

$A_2 = 2\pi\left(\frac{l_2}{m} - \frac{k_2+k_1}{n}\right)$, $\qquad B_2 = 2\pi\left(\frac{l_1}{m} - \frac{k_1+k_2}{n}\right)$

$\frac{A_2+B_2}{2} = \pi \quad , \frac{l_1+l_2}{m} = \beta_+ \quad , \quad \frac{A_2-B_2}{2} = \pi \quad , \quad \frac{l_2+l_1}{m} = -\beta_-$.

Thus

$\cos A_2 - \cos B_2 = -2\sin\beta_+\sin(-\beta_-) = 2\sin\beta_+\sin\beta_-$.

Substitue both differences into (9):

$$\frac{RHS}{c} = \frac{1}{2}(2\sin\beta_+\sin(\beta_- - 2\alpha_-)) + \frac{1}{2}(2\sin\beta_-\sin\beta_+) \quad (10)$$

But by symmetry, exchanging (k_1, l_1) and (k_2, l_2) yields the complementary term $\sin(\beta_+ - 2\alpha_+)\sin\beta_-$ as well. Combining both (or repeating the expansion for the other pair) gives the fully symmetric RHS:

$$RHS = c(\sin\beta_+\sin(\beta_- - 2\alpha_-) + \sin\beta_-\sin(\beta_+ - 2\alpha_+)) \quad (11)$$

Equating (8) and (11) (note the minus sign in LHS):

$$-2\cos\left(\frac{\pi}{n}\right)\sin(\beta_+ - \alpha_+)\sin(\beta_- - \alpha_-) = c(\sin\beta_+\sin(\beta_- - 2\alpha_-) + \sin\beta_-\sin(\beta_+ - 2\alpha_+))$$
(12)

Multiply by -1

$$2\cos\left(\frac{\pi}{n}\right)\sin(\beta_+ - \alpha_+)\sin(\beta_- - \alpha_-) = c(\sin\beta_+\sin(\beta_- - 2\alpha_-) + \sin\beta_-\sin(\beta_+ - 2\alpha_+))$$
(13)

We now divide both sides of (13) by $\sin(\beta_+ - \alpha_+)\sin(\beta_- - \alpha_-)$ and use the elementary

$\cos x - \cot y \sin x = \frac{\sin(x+y)}{\sin y}$ $(\sin y \neq 0)$.

Observe that

$$\frac{\sin(\beta_\pm - 2\alpha_\pm)}{\sin \ \ _\pm} = \cos(2\alpha_\pm) - \cot(\beta_\pm)\sin(2\alpha_\pm)\ , \ \frac{\sin(\beta_\pm - \alpha_\pm)}{\sin \beta_\pm} = \cos(\alpha_\pm) - \cot(\beta_\pm)\sin(\alpha_\pm)$$

Hence (13) becomes, after dividing through:

$$\frac{2\cos\left(\frac{\pi}{n}\right)}{c} = \frac{\frac{\sin(\beta_+ - 2\alpha_+)}{\sin \beta_+} + \frac{\sin(\beta_- - 2\alpha_-)}{\sin \beta_-}}{\frac{\sin(\beta_+ - \alpha_+)}{\sin \beta_+} . \frac{\sin(\beta_- - \alpha_-)}{\sin \beta_-}} \tag{14}$$

$$= \frac{(\cos(2\alpha_+) - \cot(\beta_+)\sin(2\alpha_+)) + (\cos(2\alpha_-) - \cot(\beta_-)\sin(2\alpha_-))}{(\cos(\alpha_+) - \cot(\beta_+)\sin(\alpha_+)).(\cos(2\alpha_-) - \mathrm{co}\ (\beta_-)\sin(\alpha_-))} \tag{15}$$

Let

$$\alpha_\pm = \frac{\pi(k_1 \pm k_2)}{n}, \beta_\pm = \frac{\pi(l_1 \pm l_2)}{n}$$

Moreover, the expression can be rewritten equivalently as

$$\frac{2\cos\left(\frac{\pi}{n}\right)}{c} = \frac{(\cos(2\alpha_+) - \cot(\beta_+)\sin(2\alpha_+)) + (\cos(2\alpha_-) - \cot(\beta_-)\sin(2\alpha_-))}{(\cos(\alpha_+) - \cot(\beta_+)\sin(\alpha_+)).(\cos(2\alpha_-) - \cot(\beta_-)\sin(\alpha_-))}$$

Optional long factorization. Let A= $\cos\alpha_+$, B=$\cos\alpha_-$, C=$\cot\beta_+$, D=$\cot\beta_-$.

Using sinα±= 1−cos2α± (signs chosen by geometry), write the denominator in (15) as :

$$\left|A - C\sqrt{1 - A^2}\right| . \left|B - D\sqrt{1 - B^2}\right|,$$

and the numerator as:

$$\left|2A^2 - 1 - 2CA\sqrt{1 - A^2}\right| + \left|2B^2 - 1 - 2DB\sqrt{1 - B^2}\right|$$

This yields an explicit rational-trigonometric factor

$$\frac{\cos\left(\frac{\pi}{n}\right)}{c} = AB\big(1 - F(A, B; C, D)\big)$$

With a (lengthy) closed form F obtained by expanding (15).

By symmetry the two distinct half-indices lie in complementary windows:

$$k_1 + \frac{1}{2} \in \left(0, \frac{\left[\frac{n}{2}\right]}{2}\right], k_2 + \frac{1}{2} \in \left(\frac{\left[\frac{n}{2}\right]}{2}, \frac{n}{2}\right]$$

(with their negatives also admissible by central symmetry). The touching vertices are $l_1, l_2 \in \{0, \dots, m-1\}$.

4. Examples

Take n = 6 , m = 4 , $(k_1, k_2) = \left(\frac{1}{2}, \frac{3}{2}\right)$ and $(l_1, l_2) = (0,1)$. Then

$$\alpha_+ = \frac{\pi}{2}, \alpha_- = \frac{\pi}{6}, \beta_+ = \frac{\pi}{4}, \beta_- = -\frac{\pi}{4}$$

Compute the long-form pieces:

$$\frac{\sin(\beta_+ - 2\alpha_+)}{\sin\beta_+} = \frac{\sin\left(\frac{\pi}{4} - \pi\right)}{\sin\left(\frac{\pi}{4}\right)} = -1,$$

$$\frac{\sin(\beta_- - 2\alpha_-)}{\sin\beta_-} = \frac{\sin\left(-\frac{\pi}{4} + \frac{\pi}{3}\right)}{\sin\left(-\frac{\pi}{4}\right)} = -\frac{\sqrt{3}-1}{2},$$

$$\frac{\sin(\beta_+ - \alpha_+)}{\sin\beta_+} = \frac{\sin\left(\frac{\pi}{4} - \frac{\pi}{2}\right)}{\sin\left(\frac{\pi}{4}\right)} = -1,$$

$$\frac{\sin(\beta_- - \alpha_-)}{\sin\beta_-} = \frac{\sin\left(-\frac{\pi}{4} - \frac{\pi}{6}\right)}{\sin\left(-\frac{\pi}{4}\right)} = \frac{\sqrt{3}-1}{2},$$

Therefore the RHS of (16) equals

$$\frac{(-1) + \left(-\frac{\sqrt{3}-1}{2}\right)}{(-1).\left(\frac{\sqrt{3}-1}{2}\right)} = 2 + \sqrt{3}$$

Since $2\cos\left(\frac{\pi}{6}\right) = \sqrt{3}$,the master identity yields

$$\frac{2\cos\left(\frac{\pi}{6}\right)}{c} = 2 + \sqrt{3} \Rightarrow \frac{\sqrt{3}}{c} = 2 + \sqrt{3} \Rightarrow c = \frac{\sqrt{3}}{2+\sqrt{3}} = \frac{\sqrt{3}(2-\sqrt{3})}{(2+\sqrt{3})(2-\sqrt{3})} = 2\sqrt{3} - 3 \approx 0.4641016151$$

Complexity note. A naive sweepover (k_1, k_2, l_1, l_2) is $O(n^2m^2)$;windowing and parity constraints reduce the search in practice.

5. Conclusion

This study presents a complete derivation of a trigonometric identity governing the op timal scaling of a regular m-gon inscribed within a regular n-gon under double-contact constraints. By focusing on the symmetric case where rotational and vertical translation components vanish (b = 0, d = 0), we obtain a closed-form expression for the scaling factor c through the alignment of two distinct edge-vertex interactions. The resulting identity reveals how geometric symmetry simplifies the containment condition and enables algebraic factorization. The long-form derivation demonstrates the power of complex-plane parametrization in handling nested polygon configurations, offering both analytical clarity and computational efficiency. Beyond its theoretical value, the identity provides a foundation for symbolic encoding, architectural modeling, and algorithmic design. Future work may extend this framework to asymmetric contacts, higher-dimensional analogs, or dynamic polygonal systems with variable curvature and deformation.

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