Türkiye Mathematical Sciences
Year:2025 Vol: 2 Issue:2
E-ISSN : 3062-3138
Research Article

Finite Time Blow up in a Triharmonic Nonlinear Wave Model with Variable Damping

Nebi YILMAZ*[1], Erhan PİŞKİN[2]
*[1]*Dicle University, Institute of Natural and Applied Sciences, Department of Mathematics, Diyarbakır,Türkiye,*
nebiyilmaz1981@gmail.com, ORCID: 0000-0001-5615-9043
[2]Dicle University, Department of Mathematics, Diyarbakır,Türkiye,
episkin@dicle.edu.tr, ORCID:0000-0001-6587-4479

Abstract
In this work, we deal with the triharmonic wave equations. We established blow up solution with negative initial energy under suitable conditions on variable exponents.

Keywords: *Blow up, nonlinear equation, Sobolev spaces Variable exponents.*

MSC: 35B44, 35G20, 35L25

Received: 25/11/2025
Accepted: 31/12/2025

1. INTRODUCTION

1.1 Setting of the problem:

In this paper, we consider the following the initial-boundary value problem equations with variable exponents

$$\begin{cases} u_{tt} - \Delta^3_{m(x)} u + u_t + a|u_t|^{p(\cdot)-2}u_t = b|u|^{q(\cdot)-2}u, & \Omega \times (0,\infty), \\ u(x,t) = \Delta u(x,t) = \Delta^2 u(x,t) = 0, & \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & \Omega. \end{cases} \quad (1)$$

The $m(x)$ −triharmonic $\Delta^3_{m(x)}$ is the nonlinear differential operator defined by

$$\Delta^3_{m(x)} u = div\left(\Delta\left(|\nabla\Delta u|^{m(\cdot)-2}\nabla\Delta u\right)\right), \text{ for all } u \in W^{3,\,m(\cdot)}(\Omega)$$

and $m(\cdot)$, $p(\cdot)$ and $q(\cdot)$ satisfy the log-Hölder continuity condition:

$$|p(x)-p(y)| \leq -\frac{A}{ln|x-y|}, for\ all\ x,y \in \Omega, with\ |x-y| < \delta, \quad (2)$$

where $0 < \delta < 1$ and $A > 0$.

The exponents $m(\cdot)$, $p(\cdot)$ and $q(\cdot)$ are measurable functions on Ω satisfiying

$$2 \leq max\{p_1, q_2\} < m_1 \leq m(x) \leq m_2 \leq q_*(x), \quad (3)$$

with

$$\begin{cases} m_1 = ess\ inf_{x\in\Omega}\, m\,(x),\ m_2 = ess\ sup_{x\in\Omega}\, m\,(x), \\ p_1 = ess\ inf_{x\in\Omega}\, p\,(x),\ p_2 = ess\ sup_{x\in\Omega}\, p\,(x), \\ q_1 = ess\ inf_{x\in\Omega}\, q\,(x),\ q_2 = ess\ sup_{x\in\Omega}\, q\,(x), \end{cases}$$

and

$$q_*(x) = \begin{cases} \dfrac{nq(x)}{ess\ sup_{x\in\Omega}(n-q(x))} & \text{if } q_2 < n, \\ +\infty & \text{if } q_2 \geq n. \end{cases}$$

1.2 Literature Overview:

Ge et al. [17] investigated the following equation:

$$\Delta\left(|\Delta u|^{p(x)-2}\Delta u\right) = \lambda V(x)\left(|u|^{q(x)-2}u\right).$$

They demonstrated the existence of a continuous family of eigenvalues, considering various cases of growth rates involved in the problem.

Liu [21] discussed the initial boundary value problem for a $p(x)$ −fourth-order parabolic equation with nonstandard growth conditions

$$u_t + \Delta^2_{p(x)}u = |u|^{q(x)-2}u.$$

He establish the local existence of weak solutions and derive the finite time blow up of solutions with non-positive initial energy.

Messaoudi et al. [24] investigate the following problem

$$u_{tt} - div\left(|\nabla u|^{r(\cdot)-2}\nabla u\right) + |u_t|^{m(\cdot)-2}u_t = 0.$$

Under suitable assumptions on the initial value, decay estimate was proved by using a lemma
by Komornik.
Ferreira et al. [14] studied a nonlinear Petrovsky equation with a strong damping and the

$p(x)$-biharmonic term as follows

$$u_{tt} + \Delta^2_{p(x)} u - div\left(|\nabla u|^{r(\cdot)-2} \nabla u\right) + |u_t|^{m(\cdot)-2} u_t = f(x,t,u_t).$$

They used the the Faedo-Galerkin method to establish the existence of weak solution.
Belakhdar et al. [6] study the properties of the eigenvalue of the $p(x)$-triharmonic problem

$$-\Delta^3_{p(x)} u = \lambda V_1(x)\left(|u|^{q(x)-2} u\right),$$

under adequate conditions on the exponent functions p, q and the weight function V_1.
In [22], Messaoudi and Talahmeh studied the following nonlinear wave equations with variable exponents

$$u_{tt} - div\left(|\nabla u|^{r(\cdot)-2} \nabla u\right) + a|u_t|^{m(\cdot)-2} u_t = b|u|^{p(x)-2} u.$$

They obtained the blow up of solutions with negative initial energy. Also, Messaoudi et al. [23] proved the existence and stability of solutions the same system.

In [34], Yılmaz and Pişkin studied the delayed m-Laplacian wave equation

$$z_{tt} - \Delta_m z + z + \mu_1 |z_t|^{p(\cdot)-2} z_t + \mu_2 z_t(x, t-\tau) |z_t|^{p(\cdot)-2}(x, t-\tau) = z|z|^{q(\cdot)-2} \ln|z|^k.$$

Recently, problems with variable exponents have been handled carefully in several papers, some results relating the local existence, global existence, blow up and stability. Problems involving variable exponents arise in various branches of science, including image processing, electrorheological fluids, and nonlinear elasticity theory [20, 34]. Furthermore, a substantial body of research concerning the theory of partial differential equations within this framework can be found in [4, 5, 7, 8, 10, 11, 13, 15, 16, 20, 26, 27, 28, 29, 30, 31, 35, 36].
Blow up phenomena commonly arise in solutions to nonlinear partial differential equations of various types. A recent comprehensive overview of these methods can be found in the monograph by Al'shin et al. [2], Hu [19] and Pişkin [25].
Motivated by the above studies, we proved to blow up the variable-exponent triharmonic Klein-Gordon wave equations. The rest of our work is organized as follows: In section 2, we give some lemmas and theorem. In section 3, we state and prove our main result.

2. PRELIMINARIES

We introduce in this part some preliminary informations about the Lebesgue spaces and Sobolev spaces with variable exponents [1, 9, 12, 32].
Let $q: \Omega \to [1, \infty]$ be a measurable function, where Ω is a domain of R^n. We define the variable exponent Lebesgue space by

$$L^{q(x)}(\Omega) = \left\{u : \Omega \to R;\ u \text{ measurable in } \Omega : \varrho_{q(\cdot)}(\lambda u) < \infty, \text{for some } \lambda > 0\right\},$$

where

$$\varrho_{q(\cdot)}(u) = \int_{\Omega} \frac{1}{q(x)} |u(x)|^{q(x)}\, dx$$

is a modular. Equipped with the following Luxembourg-type norm

$$\|u\|_{q(\cdot)} = inf \left\{ \lambda > 0 : \int_{\Omega} \left| \frac{u(x)}{\lambda} \right|^{q(x)} dx \le 1 \right\},$$

$L^{q(\cdot)}(\Omega)$ is a Banach space.

Next, we define the variable-exponent Sobolev space $W^{m,p(\cdot)}(\Omega)$ as

$$W^{m,q(\cdot)}(\Omega) = \left\{ u \in L^{q(\cdot)}(\Omega) \ : \ D^{\alpha} u \in L^{q(\cdot)}(\Omega),\ |\alpha| \le m \right\}.$$

Lemma 1. *[9] Let* $\Omega \subset R^n$ *be a bounded domain and suppose that* $q(\cdot)$ *satisfies (20), then*

$\|u\|_{q(\cdot)} \le C \|\nabla u\|_{q(\cdot)}$, for all $u \in W_0^{1,q(\cdot)}(\Omega)$,

where $C = C(q_1, q_2, \Omega) > 0$. *In particular,* $\|\nabla u\|_{q(\cdot)}$ *defines an equivalent norm on* $W_0^{1,q(\cdot)}(\Omega)$.

Similar to the Lemma above, the following inequality can be written for $u \in W_0^{3,q(\cdot)}(\Omega)$:

$\|u\|_{q(\cdot)} \le C \|\nabla \Delta u\|_{q(\cdot)}$.

Lemma 2. *[9] If* $m(\cdot) \in C(\overline{\Omega})$ *and* $r : \Omega \to [1, \infty)$ *is a measurable function such that*

$$ess\ inf_{x \in \Omega} \big(m^*(x) - r(x)\big) > 0 \text{ with } m^*(x) = \begin{cases} \dfrac{nm(x)}{ess\ sup_{x\Omega}\big(n - m(x)\big)} & \text{if } m_2 < n, \\ \infty & \text{if } m_2 \ge n. \end{cases}$$

Then the embedding $W_0^{1,m(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$ *is continuous and compact.*

Lemma 3. *[9] Let* $p,\ q,\ s \ge 1$ *are measurable functions defined on* Ω *such that*

$$\frac{1}{s(l)} = \frac{1}{p(l)} + \frac{1}{q(l)}, \text{ for a.e. } l \in \Omega.$$

If $u \in L^{p(\cdot)}(\Omega)$ and $L^{q(\cdot)}(\Omega)$, then $uv \in L^{s(\cdot)}(\Omega)$, with

$\|uv\|_{s(\cdot)} \le 2 \|u\|_{p(\cdot)} \|v\|_{q(\cdot)}$.

Lemma 4. *[9] If q is a measurable function on Ω satisfying (25), then we have*

$$min\left\{\|u\|_{q(\cdot)}^{q_1}, \|u\|_{q(\cdot)}^{q_2}\right\} \leq \varrho_{q(\cdot)}(u) \leq max\left\{\|u\|_{q(\cdot)}^{q_1}, \|u\|_{q(\cdot)}^{q_2}\right\},$$

for any $u \in L^{q(\cdot)}(\Omega)$.

The proof of the following proposition can be established employing the Galerkin method as in the work of Antontsev [3].

Theorem 5. *Let $u_0 \in W_0^{3,m(\cdot)}(\Omega)$, $u_1 \in L^2(\Omega)$ and assume that the exponents m, p, q satisfy conditions (25) and (20). Then problem (10) has a unique weak solution such that*

$$\begin{cases} u \in L^\infty\left((0,T), W_0^{3,m(\cdot)}(\Omega)\right), \\ u_t \in L^\infty\left((0,T), L^2(\Omega)\right), \\ u_{tt} \in L^\infty\left((0,T), W_0^{-3,m'(\cdot)}(\Omega)\right), \end{cases}$$

where $\frac{1}{m(\cdot)} + \frac{1}{m'(\cdot)} = 1.$

3. BLOW UP

In this section we aim to prove Theorem 13. Therefore we first introduce the corresponding energy functional

$$E(t) = \frac{1}{2}\int_\Omega |u_t|^2\, dx + \int_\Omega \frac{1}{m(x)} |\nabla \Delta u|^{m(x)}\, dx - b\int_\Omega \frac{1}{q(x)} |u|^{q(x)}\, dx. \tag{4}$$

Lemma 6. *Suppose the conditions of Lemma 2 hold. Then, there exists a constant $C > 1$, which depends on Ω only, such that*

$$\varrho_{q(x)}^{\frac{s}{q_1}}(u) \leq C\left(\|\nabla \Delta u\|_{m(\cdot)}^{m_1} + \varrho_{q(\cdot)}(u)\right) \tag{5}$$

for any $u \in W_0^{3,m(\cdot)}(\Omega)$ and $m_1 \leq s \leq q_1$.

Proof of Lemma 6. If $\varrho_{q(\cdot)}(u) > 1$, we deduce that

$$\varrho_{q(\cdot)}^{\frac{s}{q_1}}(u) \leq \varrho_{q(\cdot)}(u) \leq C\left(\|\nabla \Delta u\|_{m(\cdot)}^{m_1} + \varrho_{q(\cdot)}(u)\right).$$

On the other hand, if $\varrho_{q(\cdot)}(u) \leq 1$, then Lemma 3 ensures, $\|u\|_{q(\cdot)} \leq 1$. Making use of Lemmas 2 and 4, we arrive at

$$\varrho_{q(\cdot)}^{\frac{s}{q_1}}(u) \le \varrho_{q(\cdot)}^{\frac{m_1}{q_1}}(u) \le \left[max\left\{\|u\|_{m(\cdot)}^{q_1}, \|u\|_{m(\cdot)}^{q_2}\right\}\right]^{\frac{m_1}{q_1}} \le \|u\|_{q(\cdot)}^{m_1} \le C\|\nabla\Delta u\|_{m(\cdot)}^{m_1}$$

where $C > 1$. As a result, (5) is established. ■

As a particular case, we obtain the following lemmas.

Lemma 7. *Under the assumptions of Lemma 6, we have*

$$\|u\|_{q_1}^{s} \le C\left(\|\nabla\Delta u\|_{m_1}^{m_1} + \|u\|_{q_1}^{q_1}\right) \quad (6)$$

for any $u \in W_0^{3,m(\cdot)}(\Omega)$ *an* $m_1 \le s \le q_1$.

Now, we let

$$H(t) = -E(t),$$

where C a positive constant. Combination of (4) and (5) leads to the following.

Lemma 8. *Under the assumptions of Lemma 6, we have*

$$\varrho_{r(\cdot)}^{\frac{s}{q_1}}(u) \le C\left(|H(t)| + \|u_t\|_2^2 + \varrho_{q(\cdot)}(u)\right), \quad (7)$$

for any $u \in W_0^{3,m(\cdot)}(\Omega)$ *and* $m_1 \le s \le q_1$.

Lemma 9. *Under the assumptions of Lemma 6, we have*

$$\|u\|_{r_1}^{s} \le C\left(|H(t)| + \|u_t\|_2^2 + \|u\|_{q_1}^{q_1}\right) \quad (8)$$

for any $u \in W_0^{3,m(\cdot)}(\Omega)$ *and* $m_1 \le s \le q_1$.

Lemma 10. *Assume that (2) and (3) hold and* $E(0) < 0$. *Then the solution of (10) satisfies, for some* $c > 0$,

$$\varrho_{q(\cdot)}(u) \ge c\|u\|_{q_1}^{q_1}. \quad (9)$$

Lemma 11 *Suppose that (3) holds and let* u *be the solution of (10). Then,*

$$\int_\Omega |u|^{p(x)}\,dx \le C\left(\left(\varrho_{r(\cdot)}(u)\right)^{\frac{p_1}{q_1}} + \left(\varrho_{p(\cdot)}(u)\right)^{\frac{p_2}{q_1}}\right). \quad (10)$$

Lemma 12. *Le* u *be the solution of (1). Then there exists a constant* $c_1 > 0$ *such that*

$$\|\nabla\Delta u(\cdot, t_k)\|_{m(\cdot)} \ge c_{1,}\ \forall t \ge 0.. \quad (11)$$

Proof of Lemma 12. Assume, for the sake of contradiction, that there exists a sequence t_k such that

$$\|\nabla\Delta u(\cdot,t_k)\|_{m(\cdot)} \to 0 \text{ as } k \to \infty.$$

By employing Lemmas 2 and 4, we deduce that

$$\varrho_{r(\cdot)}(\cdot,t_k) \to 0 \text{ as } k \to \infty.$$

Consequently,

$$E(t_k) \geq 0$$

which conflicts with the assertion that $E(t_k) < 0,\ \forall t \geq 0$.
Now we state our main result.

Theorem 13. *Let the assumptions of Theorem 5 be satisfied and assume that*

$$E(0) < 0. \tag{12}$$

Then the solution of (1) blows up in finite time.

Proof of Theorem 13. We multiply (1) by u_t and integrate over the domain Ω to obtain

$$E'(t) = -a\int_\Omega |u_t(x,t)|^{q(x)}\,dx \leq 0 \tag{13}$$

for a.e. $t \in [0,T)$, since $E(t)$ is an absolutely continuous function [17];

hence, $H'(t) \geq 0$ and

$$0 < H(0) \leq H(t) \leq \frac{b}{q_1}\varrho_{q(\cdot)}(u), \tag{14}$$

for all t in $[0,T)$, by taking into account (12). We define

$$\Psi(t) = H^{1-\alpha}(t) + \varepsilon\int_\Omega uu_t(x,t)\,dx, \tag{15}$$

for ε small to be chosen later and

$$0 < \alpha \leq min\left\{\frac{q_1-2}{2q_1}, \frac{q_1-p_2}{r_1(p_2-1)}\right\}. \tag{16}$$

We differentiate (15) and use the equation in (1) to arrive at

$$\Psi'(t) = (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon \int_{\Omega} |u_t|^2 \, dx - \varepsilon \int_{\Omega} |\nabla \Delta u|^{m(x)} \, dx + \varepsilon b \int_{\Omega} |u|^{q(x)} \, dx$$
$$-\varepsilon a \int_{\Omega} u|u_t|^{p(x)-2}u_t \, dx. \quad (17)$$

Applying the definition of $H(t)$, it can be inferred that

$$-\varepsilon(1-\eta)q_1 H(t) = \frac{\varepsilon(1-\eta)q_1}{2}\int_{\Omega} |u_t|^2 \, dx + \int_{\Omega} \frac{\varepsilon(1-\eta)q_1}{m(x)} |\nabla \Delta u|^{m(x)} \, dx$$
$$-b\varepsilon(1-\eta)q_1 \int_{\Omega} \frac{1}{q(x)} |u|^{q(x)} \, dx,$$

where $0 < \eta < 1$. Adding and subtracting $-\varepsilon(1-\eta)q_1 H(t)$ from the right-hand side of (17), we get

$$\Psi'(t) \geq (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon(1-\eta)q_1 H(t) + \varepsilon\left(1 + \frac{(1-\eta)q_1}{2}\right)\int_{\Omega} |u_t|^2 \, dx$$
$$+ \varepsilon\left(\frac{(1-\eta)q_1}{m_2} - 1\right)\int_{\Omega} |\nabla \Delta u|^{m(x)} \, dx + \varepsilon b\eta \int_{\Omega} |u|^{q(x)} \, dx$$
$$- \varepsilon a \int_{\Omega} u|u_t|^{p(x)-2}u_t \, dx. \quad (18)$$

Then, for η small enough, we have

$$\Psi'(t) \geq (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon\beta\left[H(t) + \int_{\Omega} |u_t|^2 \, dx + \int_{\Omega} |u|^{q(x)} \, dx + \int_{\Omega} |\nabla \Delta u|^{m(x)} \, dx\right]$$
$$- \varepsilon \int_{\Omega} u|u_t|^{p(x)-2}u_t \, dx, \quad (19)$$

where

$$\beta = min\left\{(1-\eta)q_1, b\eta, \left(1 + \frac{(1-\eta)q_1}{2}\right), \left(\frac{(1-\eta)q_1}{m_2} - 1\right)\right\} > 0.$$

To estimate the last term in (19), we use the Young inequality, we have

$$\int_{\Omega} |u_t|^{p(x)-1}|u| \, dx \leq \frac{1}{p_1}\delta^{p(x)}|u|^{p(x)} + \frac{p_2-1}{p_2}\int_{\Omega} \delta^{-\frac{p(x)}{p(x)-1}}|u_t|^{p(x)} \, dx, \quad (20)$$

where $\delta > 0$ is constants depending on the time t and specified later. Let us choose δ so that

$$\delta^{-\frac{p(x)}{p(x)-1}} = kH^{-\alpha}(t),$$

for a large constant k to be specified later, and substituting in (20), we get

$$\int_\Omega |u_t|^{p(x)-1}|u|\,dx \le \frac{1}{p_1}\int_\Omega k^{1-p(x)}|u|^{p(x)}H^{\alpha(p(x)-1)}(t)\,dx + \frac{p_2-1}{ap_2}kH^{-\alpha}(t)H'(t). \tag{21}$$

Combining (19) and (20) gives

$$\Psi'(t) \ge \left[(1-\alpha) - \varepsilon\left(\frac{p_2-1}{p_2}\right)k\right]H^{-\alpha}(t)H'(t) + \varepsilon\beta\left[H(t) + \int_\Omega |u_t|^2\,dx + \int_\Omega |\nabla\Delta u|^{m(x)}\,dx\right]$$
$$-\frac{\varepsilon k^{1-p_1}a}{p_1}CH^{\alpha(p_2-1)}(t)\int_\Omega |u|^{p(x)}\,dx. \tag{22}$$

By using (14) and Lemma 11, we have

$$H^{\alpha(p_2-1)}(t)\int_\Omega |u|^{p(x)}\,dx \le C_1\left[\left(\varrho_{r(\cdot)}(u)\right)^{\frac{p_1}{q_1}+\alpha(p_2-1)} + \left(\varrho_{r(\cdot)}(u)\right)^{\frac{p_2}{q_1}+\alpha(p_2-1)}\right]. \tag{23}$$

We then use (16) and Lemma 6, for

$s = p_1 + \alpha q_1(p_2-1) \le q_1$, *and* $s = p_2 + \alpha q_1(p_2-1) \le q_1$,

to do deduce, from (23), that

$$H^{\alpha(p_2-1)}(t)\int_\Omega |u|^{p(x)}\,dx \le C\left[\|\nabla\Delta u\|_{m(\cdot)}^{m_1} + \varrho_{q(\cdot)}(u)\right]. \tag{24}$$

By exploiting Lemmas 4 and 12, we get

$$\varrho_{m(\cdot)}(\nabla\Delta u) \ge c_2\|\nabla\Delta u\|_{m(\cdot)}^{m_1}. \tag{25}$$

Combining (22), (24) and (25) leads to

$$\Psi'(t) \ge \left[(1-\alpha) - \varepsilon\frac{p_2-1}{p_2}k\right]H^{-\alpha}(t)H'(t)$$
$$+\varepsilon\left(\beta - \frac{k^{1-p_1}}{p_1}aC\right)\left[H(t) + \|u_t\|^2 + \varrho_{m(\cdot)}(\nabla\Delta u) + \varrho_{q(\cdot)}(u)\right]. \tag{26}$$

Now, we select k so large that $\gamma = \beta - a\frac{k^{-q_1}}{q_1}C > 0$.
Once k is chosen (hence γ), we select ε so small that
$(1-\alpha) - \varepsilon\dfrac{q_2-1}{q_2}k \ge 0$

and

$$\Psi(0) = H^{1-\alpha}(0) + \varepsilon \int_{\Omega} u_0(x)u_1(x)\,dx > 0.$$

Hence (26) takes the form

$$\Psi'(t) \geq \gamma\varepsilon\left[H(t) + \|u_t\|^2 + \varrho_{m(\cdot)}(\nabla\Delta u) + \varrho_{q(\cdot)}(u)\right] \geq \gamma\varepsilon\left[H(t) + \|u_t\|^2 + \|u\|_{q_1}^{q_1}\right], \quad (27)$$

by virtue of (9). Consequently, we get

$$\Psi(t) \geq \Psi(0) > 0, for\ all\ t \geq 0.$$

Next, we want to obtain an inequality of the form

$$\Psi'(t) \geq \xi\Psi^{\frac{1}{1-\sigma}}(t), \text{ for all } t \geq 0, \quad (28)$$

for a $\xi = (\varepsilon\gamma,\ C) > 0$, (here C is the constant of Lemma 7). Once (28) is proved, one can obtain, in a standard way, the finite-time blowup of the functional $\Psi(t)$.
To prove (28), we first note that

$$\left|\int_{\Omega} uu_t\,dx\right| \leq \|u\|_2\|u_t\|_2 \leq C\|u\|_{q_1}\|u_t\|_2,$$

which implies

$$\left|\int_{\Omega} uu_t dx\right|^{\frac{1}{1-\alpha}} \leq C\|u\|_{2(p_1+2)}^{\frac{1}{1-\alpha}}\|u_t\|^{\frac{1}{1-\alpha}}.$$

Young's inequality gives

$$\left|\int_{\Omega} uu_t\,dx\right|^{\frac{1}{1-\alpha}} \leq C\left[\|u\|_{r_1}^{\frac{\mu}{1-\alpha}}\|u_t\|^{\frac{\theta}{1-\alpha}}\right], \quad (29)$$

for $\frac{1}{\mu} + \frac{1}{\theta} = 1$. We take $\theta = 2(1-\alpha)$, to get $\frac{\mu}{1-\alpha} = \frac{2}{1-2\alpha} \leq p_1$, by (23).

Therefore (29), becomes

$$\left|\int_{\Omega} uu_t\,dx\right|^{\frac{1}{1-\alpha}} \leq C\left[\|u\|_{2(p_1+2)}^{s} + \|u_t\|^2\right],$$

where $s = \frac{2}{1-2\alpha} \leq r_1$.

By recalling Lemma 8, we have

$$\left|\int_\Omega uu_t\,dx\right|^{\frac{1}{1-\alpha}} \leq C\left[H(t) + \|u\|_{p_1}^{p_1} + \|u_t\|^2\right], \tag{30}$$

for all $t \geq 0$. Thus,

$$\Psi^{\frac{1}{1-\alpha}}(t) = \left[H^{1-\alpha}(t) + \varepsilon \int_\Omega uu_t\,dx\right]^{\frac{1}{1-\alpha}} \leq 2^{\frac{1}{1-\alpha}}\left[H(t) + \left|\int_\Omega uu_t\,dx\right|^{\frac{1}{1-\alpha}}\right]$$
$$\leq C\left[H(t) + \|u_t\|^2 + \varrho_{m(\cdot)}(\nabla\Delta u) + \varrho(u)\right],$$

and combine with (27) and (30), the inequality (28) is established.
This completes the proof. ■

4. REFERENCES

[1] Adams, R. A., & Fournier, J. J. F. (2003). *Sobolev spaces*. Academic Press, New York.

[2] Al'shin, A., Korpusov, M. O., & Sveshnikov, A. G. (2011). *Blow-up in nonlinear Sobolev-type equations*. De Gruyter Series in Nonlinear Analysis and Applications, Berlin-New York.

[3] Antontsev, S. N. (2011). Wave equation with *p(x,t)*-Laplacian and damping term: Existence and blow-up. *Differential Equations & Applications, 3*(4), 503–525.

[4] Antontsev, S. N., Ferreira, J., & Pişkin, E. (2021). Existence and blow-up of solutions for a strongly damped Petrovsky equation with variable-exponent nonlinearities. *Electronic Journal of Differential Equations, 2021*(1), 1–18.

[5] Antontsev, S. N., Ferreira, J., Pişkin, E., & Cordeiro, S. M. S. (2021). Existence and non-existence of solutions for Timoshenko-type equations with variable exponents. *Nonlinear Analysis: Real World Applications, 61*, 103341,1-13.

[6] Belakhdar, A., Belaouidel, H., Filali, M., & Tsouli, N. (2022). Existence and multiplicity of solutions of *p(x)*-triharmonic problem. *Nonlinear Functional Analysis and Applications, 27*(2), 349–361.

[7] Butakın, G., & Pişkin, E. (2024). Existence and blow-up of solutions for m(x)-biharmonic equation with variable exponent sources. *Filomat, 38*(22), 7871–7893.

[8] Cömert, T., & Pişkin, E. (2021). Blow-up at infinity of weak solutions for a higher-order parabolic equation with logarithmic nonlinearity. *Journal of Universal Mathematics, 4*(2), 118–127.

[9] Diening, L., Harjulehto, P., Hästö, P., & Růžička, M. (2011). *Lebesgue and Sobolev spaces with variable exponents*. Springer.

[10] Ekinci, F., & Pişkin, E. (2021). Blow-up and exponential growth for a Petrovsky equation with degenerate damping. *Universal Journal of Mathematics and Applications, 4*(2),

82–87.

[11] Ekinci, F., & Pişkin, E. (2023). Blow-up and exponential growth for a Kirchhoff-type viscoelastic equation with degenerate damping term. *Mathematical Sciences and Applications E-Notes, 11*(3), 153–163.

[12] Fan, X., & Zhao, D. (2001). On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$ *Journal of Mathematical Analysis and Applications, 263*(2), 424–446.

[13] Ferreira, J., Panni, W. S., Messaoudi, S. A., Pişkin, E., & Shahrouzi, M. (2022). Existence and asymptotic behavior of beam-equation solutions with strong damping and *p(x)*-biharmonic operator. *Journal of Mathematical Physics, Analysis, Geometry, 18*(4), 488–513.

[14] Ferreira, J., Pişkin, E., & Shahrouzi, M. (2024). General decay and blow-up of solutions for a plate viscoelastic *p(x)* Kirchhoff type equation with variable exponent nonlinearities and boundary feedback. *Quaestiones Mathematicae, 47*(4), 813–830.

[15] Ferreira, J., Pişkin, E., Raposo, C., Shahrouzi, M., & Yüksekkaya, H. (2022). Stability result for a Kirchhoff beam equation with variable exponent and time delay. *Universal Journal of Mathematics and Applications, 5*(1), 1–9.

[16] Fidan, A., & Pişkin, E. (2024). Decay of solutions for a singular parabolic mmm-biharmonic equation with logarithmic nonlinearity. *Türkiye Mathematical Sciences*, 1–11.

[17] Ge, B., Zhou, Q. M., & Wu, Y. H. (2015). Eigenvalue of the *p(x)*-biharmonic operator with indefinite weight. *Zeitschrift für Angewandte Mathematik und Physik, 66*, 1007–1021.

[18] Georgiev, V., & Todorova, G. (1994). Existence of solutions of the wave equation with nonlinear damping and source terms. *Journal of Differential Equations, 109*(2), 295–308.

[19] Hu, B. (2011). *Blow-up theories for semilinear parabolic equations* (Lecture Notes in Mathematics, Vol. 2018). Springer.

[20] Irkıl, N., & Pişkin, E. (2021). Global nonexistence of solutions for the higher order Kirchhoff type system with logarithmic nonlinearities. *Journal of Universal Mathematics, 4*(2), 172–187.

[21] Liu, Y. (2019). Existence and blow-up of solutions to a parabolic equation with nonstandard growth conditions. *Bulletin of the Australian Mathematical Society, 99*(2), 242–249.

[22] Messaoudi, S. A., & Talahmeh, A. A. (2017). Blow-up in solutions of a quasilinear wave equation with variable-exponent nonlinearities. *Mathematical Methods in the Applied Sciences, 40*(18), 6976–6986.

[23] Messaoudi, S. A., Talahmeh, A. A., & Al-Gharabli, M. M. (2022). On the existence and stability of a nonlinear wave system with variable exponents. *Asymptotic Analysis, 128*(2), 211–238.

[24] Messaoudi, S. A., Talahmeh, A. A., & Al-Smail, J. (2018). Decay for solutions of a nonlinear damped wave equation with variable-exponent nonlinearities. Computers & Mathematics with Applications, 76(8), 1863–1875.

[25] Pişkin, E. (2022). *Blow-up of solutions of evolution equations*. Pegem Publishing.

[26] Pişkin, E., & Altındağ, Ş. (2019). Exponential growth of solutions for nonlinear coupled viscoelastic wave equations. *Universal Journal of Mathematics and Applications, 2*(2), 70–78.

[27] Pişkin, E., & Butakın, G. (2023). Existence and decay of solutions for a parabolic-type Kirchhoff equation with variable exponents. Journal of Mathematical Sciences and Modelling, 6(1), 32–41.

[28] Pişkin, E., Dinç, Y., & Tunç, C. (2020). Lower bounds for the blow-up time to a coupled nonlinear hyperbolic type equations. Communications in Advanced Mathematical Sciences, 3(1), 53–56.

[29] Pişkin, E., & Ekinci, F. (2020). Blow-up and growth of solutions for a parabolic type Kirchhoff equation with multiple nonlinearities. Konuralp Journal of Mathematics, 8(1), 216–222.

[30] Pişkin, E., & Fidan, A. (2022). Nonexistence of global solutions for the strongly damped wave equation with variable coefficients. Universal Journal of Mathematics and Applications, 5(2), 51–56.

[31] Pişkin, E., & Yüksekkaya, H. (2021). Nonexistence of global solutions of a delayed wave equation with variable exponents. Miskolc Mathematical Notes, 22(2), 841–859.

[32] Pişkin, E., & Okutmuştur, B. (2021). *An introduction to Sobolev spaces*. Bentham Science.

[33] Růžička, M. (2000). *Electrorheological fluids: Modeling and mathematical theory*. Springer.

[34] Yılmaz, N., & Pişkin, E. (2024). Global existence and decay of solutions for a delayed mmm-Laplacian equation with logarithmic term in variable-exponent Sobolev spaces. *Filomat, 38*(23), 8157–8168.

[35] Yılmaz, N., Demir, M., & Pişkin, E. (2026). Local existence and blow-up of solutions for the higher-order viscoelastic equation with general source term and variable exponents: Theoretical and numerical results. *Nonlinear Analysis: Real World Applications, 89*, 104512.

[36] Yüksekkaya, H., & Pişkin, E. (2021). Blow-up results for a viscoelastic plate equation with distributed delay. *Journal of Universal Mathematics, 4*(2), 128–139.