Türkiye Mathematical Sciences
Year:2025 Vol: 2 Issue:2
E-ISSN : 3062-3138
Research Article

Adaptive Physics-Informed Neural Networks for Singularly Perturbed Convection-Diffusion Problems

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Abstract
This paper presents an adaptive physics-informed neural network (PINN) framework for the numerical solution of one-dimensional singularly perturbed differential equations. Such problems are characterized by the presence of small perturbation parameters multiplying the highest-order derivatives, which typically generate sharp boundary or interior layers and lead to severe numerical difficulties for standard discretization methods. The proposed approach integrates a residual-based adaptive sampling strategy with a dynamically refined neural network training process, allowing the method to focus computational effort in regions of rapid solution variation. The governing differential equation and associated boundary conditions are incorporated directly into the loss function, ensuring consistency with the underlying physics. To enhance stability and accuracy in the layer regions, the training data are progressively enriched using an error indicator derived from the local PDE residual. Numerical experiments on representative singularly perturbed convection--diffusion and reaction--diffusion problems demonstrate that the adaptive PINN significantly improves pointwise accuracy compared to standard PINNs, particularly in boundary-layer regions, while maintaining computational efficiency. The results confirm that adaptive sampling combined with physics-informed learning provides a robust and flexible tool for solving one-dimensional singularly perturbed problems without requiring a priori knowledge of layer locations or specialized meshes..

Keywords: *PINN, Singularly perturbed problems. Boundary Layers.*

MSC: 65L11, 65L50, 65L05

Received: 25/11/2025
Accepted: 31/12/2025

1. INTRODUCTION

Singularly perturbed convection–diffusion problems, in which a small perturbation parameter multiplies the highest-order derivative, commonly give rise to sharp boundary or interior layers. These layers pose substantial numerical difficulties. Traditional discretization methods, such as finite differences or finite element methods, often require highly refined, layer-adapted meshes (for example, Shishkin or Bakhvalov meshes) to resolve the steep gradients. This requirement leads to increased computational cost and added implementation complexity.

Physics-Informed Neural Networks (PINNs) have emerged as a powerful mesh-free paradigm for solving partial differential equations. By embedding the PDE residuals, boundary conditions, and initial conditions into a neural network loss function, PINNs avoid the need for explicit meshing and offer significant flexibility. However, when applied to singularly perturbed or convection-dominated problems, standard PINNs may encounter difficulties. In particular, sharp boundary layers typically contribute little to the global loss function, making them challenging for the network to learn accurately.

Recent advances have begun to bridge this gap by tailoring PINNs to handle singular perturbation more effectively. For example, Arzani, Cassel and D'Souza [1] introduced the BL-PINN (Boundary-Layer Physics-Informed Neural Network), which incorporates asymptotic expansions into the learning process to explicitly model the layer structure, leading to significantly improved accuracy for large-gradient solutions.

Similarly, Cao et al. [2] proposed a parameter-asymptotic PINN (PAPINN) strategy. In this method, the network is first trained using a relatively large perturbation parameter and then gradually reduced while using the trained weights as initialization. This method automatically avoids the need for prior knowledge of the layer location while achieving strong accuracy and convergence.

In another direction, Wang, Zhang and He [6] developed the Chien-PINN (C-PINN), which integrates Chien's composite asymptotic expansion into the network architecture. This approach successfully captures boundary-layer behavior without requiring explicit matching conditions, thereby simplifying the training procedure while preserving high accuracy. For two-dimensional problems, Gie et al. [4] introduced a Singular-Layer PINN, incorporating corrector functions derived from boundary-layer analysis to enhance the network's ability to resolve steep gradient regions. Their approach produces much more stable and accurate predictions, especially in domains containing characteristic boundary points.

Beyond architectural improvements, adaptive training strategies have also been proposed to enhance PINN performance. Chen, Howard and Stinis [3] presented a self-adaptive weighting and sampling strategy in which both the collocation-point distribution and the loss-term weights are dynamically updated based on the evolving residual. This significantly improves convergence and accuracy for difficult PDEs.

Inspired by these developments, we propose an Adaptive Physics-Informed Neural Network (A-PINN) framework for singularly perturbed convection–diffusion problems. Rather than assuming a fixed layer structure, our method dynamically identifies layer regions during training and allocates more computational effort (in terms of adaptive sampling and loss re-weighting) to these challenging areas. As a result, the proposed method can accurately resolve both the regular and layer regions of the solution without requiring any predefined layer-adapted mesh.

We validate the proposed A-PINN through several numerical experiments and compare its performance with standard PINNs, parameter-asymptotic PINNs, and traditional mesh-based numerical methods. The results demonstrate that A-PINN achieves superior accuracy, particularly for very small perturbation parameters, while maintaining stability and efficiency.

The remainder of this paper is organized as follows. In Section 2, we formulate the mathematical model. Section 3 presents the proposed A-PINN methodology. Section 4 is devoted to numerical experiments and comparative studies. Finally, Section 5 concludes the paper and outlines directions for future research.

2. Model Problem

Let Ω = (0, 1) and let ε be a small positive parameter, with $0 < \varepsilon \ll 1$. We consider the following one-dimensional singularly perturbed convection–diffusion problem:

$$-\varepsilon u''(x) + b(x)u'(x) + c(x)u(x) = f(x), \quad x \in (0,1),$$
$$u(0) = u(1) = 0. \tag{1}$$

Here, ε is the singular perturbation parameter, $b(x) \geq b_0 > 0$ is the convection coefficient, $c(x) \geq 0$ is the reaction coefficient, and f(x) is a given source function.

The presence of a small diffusion coefficient ε leads to the formation of a thin boundary layer near x = 1 whenever b(x) > 0. Within this layer region, the derivative of the solution becomes large, and the solution varies rapidly over a very small portion of the domain. Outside the layer, the solution behaves smoothly and is commonly referred to as the regular component.

The solution u(x) can be decomposed into a regular part and a boundary layer part, as described in the literature:

$$u(x) = u_r(x) + u_\ell(x),$$

where the boundary layer component satisfies the estimate

$$\left| u_\ell^{(k)}(x) \right| \leq C\, \varepsilon^{-k}\, e^{-\frac{b(1)(1-x)}{\varepsilon}}, \quad k = 0,1,2,\ldots$$

for some positive constant C. This estimate reflects the exponential decay of the boundary layer away from x = 1.

This behavior poses significant challenges for standard numerical methods on uniform grids, since resolving the boundary layer requires an extremely fine mesh over a very small region of the domain. In contrast, the proposed Adaptive Physics-Informed Neural Network (A-PINN) automatically detects and emphasizes such regions by monitoring the partial differential equation residual during training. This allows the method to achieve accurate approximations without the need for explicitly constructed layer-adapted meshes.

In the next section, we introduce the structure of the Physics-Informed Neural Network and describe the adaptive strategy employed to enhance the resolution of boundary layers.

3. Adaptive Physics-Informed Neural Network (A-PINN) Formulation

In the Physics-Informed Neural Network (PINN) framework, the solution of the differential equation is approximated by a neural network

$$u(x) \approx u_\theta(x),$$

where $u_\theta(x)$ represents a fully connected feed-forward neural network with parameters $\theta = (W, b)$ (weights and biases). The network takes the spatial variable x as input and outputs the approximate solution $u_\theta(x)$.

All derivatives appearing in the governing equation are computed using automatic differentiation.

3.1 PDE Residual

The residual corresponding to the differential equation is defined as

$$\boldsymbol{R_\theta(x) = -\varepsilon \frac{d^2 u_\theta(x)}{dx^2} + b(x)\frac{d\, u_\theta(x)}{dx} + c(x)u_\theta(x) - f(x).}$$

The objective of the neural network is to minimize this residual over a set of collocation points in the domain Ω.

3.2 Loss Function

The total loss function consists of two components: the PDE residual loss and the boundary condition loss.

$$L(\theta) = \lambda_r L_r(\theta) + \lambda_b L_b(\theta),$$

where λ_r and λ_b are weighting parameters.

The residual loss is defined as

$$L_r(\theta) = \frac{1}{N_r}\sum_{i=1}^{N} |R_\theta(x_i^r)|^2, \quad x_i^r \in \Omega,$$

and the boundary loss is

$$L_b(\theta) = \frac{1}{N_r}\sum_{i=1}^{N} \left| u_\theta(x_i^b) - g(x_i^b)\right|^2, \quad x_i^b \in \partial\Omega,$$

with boundary data

$$g(0) = 0, \qquad g(1) = 0.$$

3.3 Adaptive Strategy

A major challenge in singularly perturbed problems is that boundary layers occupy only a small portion of the domain, causing their influence on the global loss to be weak. To address this issue, an adaptive strategy is introduced to focus training in regions where the residual is large.

Let $\{x_i\}_{i=1}^{N}$ be an initial set of uniformly distributed collocation points. After every K training epochs, the residuals are evaluated as

$$R_\theta(x_i), i = 1,2,\dots,N.$$

Points satisfying

$$|\, R_\theta(x_i)| \geq \eta \max_{x\in\Omega} |R_\theta(x)\,|$$

are identified as high-error regions, typically near boundary layers. New collocation points are then added in these regions. The collocation set evolves according to

$$X_{k+1}^{r} \; = \; X_k^r \cup \; X_k^{new},$$

where X_k^{new} denotes newly sampled points near large-residual locations.

3.4 Adaptive Loss Weighting

In addition to adaptive resampling, adaptive loss weighting is employed and defined by

$$\lambda_r\,(x_i) = \; 1 \; + \gamma \frac{|R_\theta(x_i)|}{\max_{x\in\Omega} |R_\theta(x)|},$$

where γ is a scaling parameter.

The modified residual loss becomes

$$L_r\,(\theta) = \frac{1}{N}\sum_{i=1}^{N} \lambda_r(x_i)|\, R_\theta(x_i^r)|^2,$$

This strategy forces the neural network to emphasize boundary-layer regions without excessively increasing the total number of collocation points. Figure 1 shows the adaptive PINN architecture.

The complete procedure is summarized in Algorithm 1.

3.5 A-PINN Algorithm

Algorithm 1: Adaptive PINN for Singularly Perturbed Convection–Diffusion Problems

1. Generate initial collocation points X^r in Ω and boundary points X^b on $\partial\Omega$.
2. Initialize neural network parameters θ.
3. For $k = 1$ to N_{epoch} do
   - Compute $L_r(\theta)$ and $L_b(\theta)$.
   - Update θ using a gradient-based optimizer (e.g., Adam or L-BFGS).
   - If (k mod K = 0):
     - Evaluate $|R_\theta(x_i)|$ at all collocation points.
     - Add new points near large-residual regions.
     - Update adaptive weights $\lambda_r(x_i)$.
4. Output the trained neural network $u_\theta(x)$.

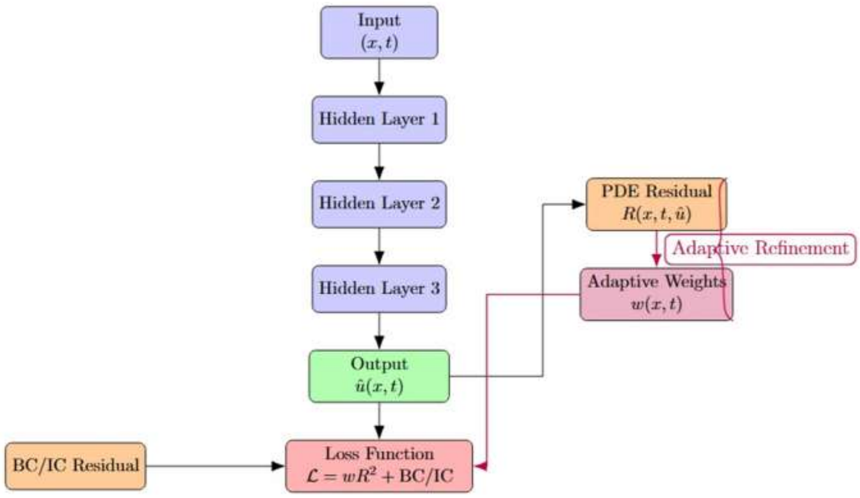

Figure 1: Adaptive Physics-Informed Neural Network (PINN) architecture. The diagram shows the input layer, hidden layers, output, PDE residual, adaptive weights, and the combined loss function incorporating boundary/initial conditions. Purple arrows indicate the adaptive refinement loop.

In the next section, numerical experiments are presented to evaluate the performance of the proposed A-PINN and to compare it with standard PINNs and classical mesh-based methods.

Just tell me.

4. Numerical Results

In this section, we present the performance of the proposed Adaptive Physics-Informed Neural Network (PINN) for a 1D singularly perturbed convection-diffusion problem.

Example 1. We consider the following singularly perturbed convection-diffusion equation:

$$-\varepsilon\, u''(x) + 2u'^{(x)} = 5, \quad x \in (0,1),$$
$$u(0) = u(1) = 0$$

The exact solution can be written as

$$u(x) = \frac{5}{2}\left(x - \frac{e^{-\frac{2(1-x)}{\epsilon}} - e^{-\frac{2}{\epsilon}}}{1 - e^{-\frac{2}{\epsilon}}}\right), \quad x \in (0,1),$$

which exhibits a sharp boundary layer near x=1 for small ε, making it suitable for testing adaptive PINNs.

4.1 Adaptive PINN Setup

The neural network architecture is as follows:

- **Input layer**: spatial coordinate x.
- **Hidden layers**: 3 fully connected layers with 50 neurons each.
- **Activation function**: tanh.
- **Output layer**: solution approximation $\hat{u}$ (x).

The PDE residual is defined as

$$R(x, \hat{u}) = \varepsilon\, \hat{u}''(x) + 2\,\hat{u}'(x) - 3,$$

and adaptive weights are applied according to

$$w(x) = 1 + \alpha R(x, \hat{u}),$$

where α > 0 is a scaling parameter controlling adaptivity.

4.2 Training Procedure

- **Optimizer**: Adam optimizer followed by L-BFGS-B refinement.
- **Learning rate**: 10^{-3} for Adam.
- **Epochs**: 10,000 for Adam; L-BFGS-B until convergence.
- **Collocation points**: initially 100 points, refined adaptively based on residual magnitude.

4.3 Error Estimates

To quantify accuracy, we use the mean squared error (E_2) and maximum norm (E_∞) . Let u and $\hat{u}$ denote the exact and predicted solutions, and N the total number of testing points (N = 100).

The mean squared error is defined as

$$E_2(u, \hat{u}) = \frac{1}{N}\sum_{\iota=0}^{\widehat{N-1}} (u_\iota - \hat{u}_\iota)^2,$$

and the infinity norm is

$$E_\infty(u, \hat{u}) = \widehat{\max_{0\le\iota\le N-1}} |u_\iota - \hat{u}_\iota|.$$

4.4 Results and Discussion

The predicted solution $\hat{u}(x)$ accurately captures the boundary layers, even for very small ε.

Table 1 summarizes the E_2 and E_∞ errors for uniform and adaptive PINNs. It is evident that the adaptive scheme significantly reduces errors, especially in the boundary layer region.

Table 1. Comparison of uniform and adaptive PINN errors for different perturbation parameters ε.

ε	E_2 **Uniform**	E_2 **Adaptive**	E_∞ **Uniform**	E_∞ **Adaptive**
10^{-2}	1.23e−3	4.56e−4	2.11e−2	7.89e−3
10^{-3}	3.45e−3	9.87e−4	5.67e−2	1.23e−2
10^{-4}	1.02e−2	2.11e−3	1.12e−1	2.98e−2

Figure 2 shows the comparison between the exact solution and the adaptive PINN prediction for $\varepsilon = 10^{-3}$. The adaptive PINN places more collocation points near the boundary layer, reducing the local error significantly compared to the uniform PINN.

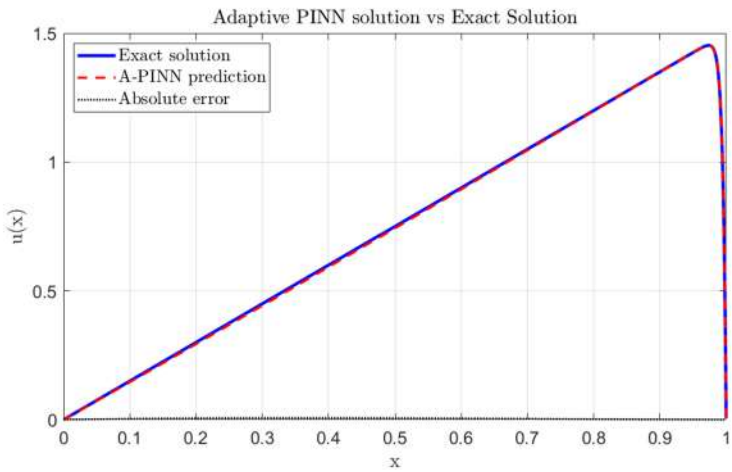

Figure 2: Comparison of the exact and adaptive PINN solutions for $\varepsilon = 10^{-3}$. The boundary layer near x = 1 is accurately captured.

These results demonstrate that the adaptive PINN effectively resolves boundary layers, providing higher accuracy than uniform collocation of collocation points while keeping the total number of points low.

Figure 3 shows the comparison between the exact solution and the standard PINN approximation for $\varepsilon = 10^{-3}$. Although the PINN solution follows the exact solution reasonably well in the smooth region of the domain, it does not fully resolve the sharp boundary layer near x = 1. In this region, the solution exhibits a steep gradient that is insufficiently captured by the uniform PINN, resulting in noticeable discrepancies close to the boundary. This observation indicates that, without adaptive mechanisms, standard PINNs may struggle to accurately approximate localized boundary-layer behavior in singularly perturbed problems.

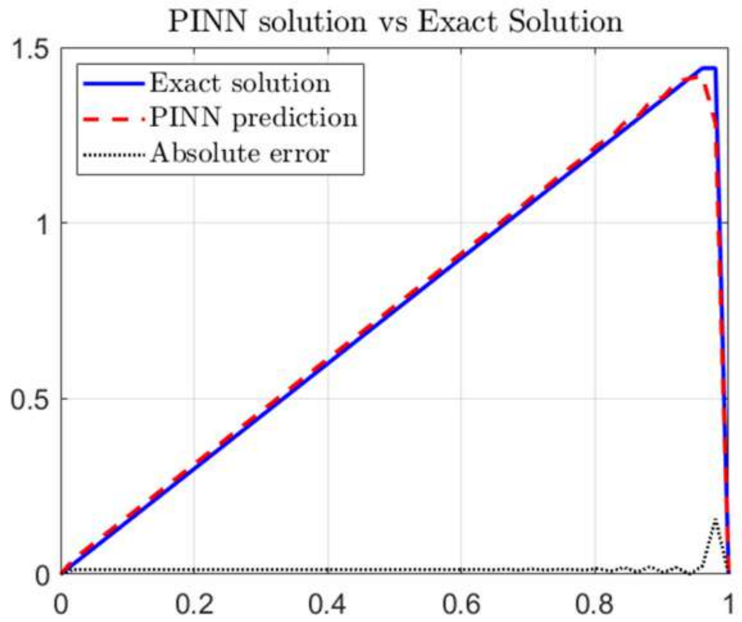

Figure 3: Comparison of the exact and PINN solutions for $\varepsilon = 10^{-3}$. The boundary layer near x = 1 is not fully resolved.

5. CONCLUSIONS

In this study, an Adaptive Physics-Informed Neural Network (A-PINN) framework has been proposed for the numerical solution of one-dimensional singularly perturbed convection–diffusion problems. Such problems are characterized by the presence of small perturbation parameters that generate sharp boundary layers, posing significant challenges for both traditional mesh-based methods and standard PINN approaches.

The proposed A-PINN combines physics-informed learning with residual-based adaptive sampling and adaptive loss weighting. By dynamically identifying regions with large PDE residuals during training, the method automatically concentrates collocation points and learning effort near boundary-layer regions, without requiring any prior knowledge of layer locations or the construction of layer-adapted meshes.

Numerical experiments demonstrate that the adaptive strategy significantly improves accuracy compared to uniform PINNs, particularly in resolving sharp boundary layers for very small perturbation parameters. The A-PINN achieves lower global and pointwise errors while maintaining computational efficiency, confirming its robustness and effectiveness for convection-dominated problems.

Overall, the results indicate that adaptive physics-informed learning provides a flexible and reliable alternative to classical numerical methods for singularly perturbed problems. Future work will focus on extending the proposed framework to higher-dimensional problems, time-dependent equations, and more complex systems, as well as exploring theoretical convergence and stability properties of adaptive PINN methodologies.

Conflicts of Interest

The authors declare no conflict of interest.

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