Journal of Tr. Math. Sci., Vol. 3, No. 3, 2026, pp. 55-75
E-ISSN: 3062-3138

Research Article

Hub Number in Ring Fuzzy Graphs and Restricted Hyper-Totient Fuzzy Graphs

Mohammed Alsharafi*[1], Haifa Ahmed[2], Saad Tobaili[3]

[1]*Corresponding author, Department of Mathematics, Faculty of Science, Sana'a University, Sana'a, Yemen.
alsharafi205010@gmail.com
[2]Department of Mathematics, Faculty of Education, Art and Science, Aden University, Aden, Yemen.
haifaahmed010@gmail.com
[3]Department of Mathematics, Faculty of Science, Hadhramout University, Mukalla, Yemen.
saadaltobaili@yahoo.com

Abstract
In this paper, we introduce and study the hub number of ring fuzzy graphs. To make the construction self-contained, a ring fuzzy graph is treated as a weighted fuzzy structure $RF\Gamma = (V, E, \mu, \sigma)$ whose underlying vertices are ring-induced substructures and whose hub number is computed by the total vertex-membership value of a minimal hub set. We investigate this parameter for several important classes of ring fuzzy graphs, including complete, path, cycle, and star ring fuzzy graphs, and obtain explicit expressions in each case. In addition, we derive general bounds for the hub number in terms of the complement graph and several neighborhood-based parameters. Relationships between the hub number and other graph-theoretic quantities, such as domination-type and chromatic-type parameters, are also examined. Furthermore, we extend the study to restricted hyper-totient fuzzy graphs and present a characterization of their hub number. Finally, we analyze the behavior of the hub number under several operations on ring fuzzy graphs, including intersection, union, direct product, and direct sum, stating the required compatibility assumptions on the underlying vertex sets and membership functions. The results obtained in this work show that the hub number is a useful structural parameter for the analysis of ring fuzzy graphs and provide a foundation for further investigations in algebraic fuzzy graph theory.

Keywords: Ring fuzzy graph; Hub set; Hub number; Restricted hyper-totient fuzzy graph; Graph operations; Algebraic fuzzy graph

MSC: 05C72, 05C69, 05C76, 05C25

Received: 10/04/2026 **Accepted:** 02/06/2026

1 Introduction

The theory of fuzzy sets, introduced by Zadeh [1], has become one of the fundamental mathematical tools for modeling uncertainty and partial membership in complex systems. This idea was extended to graph theory by Rosenfeld [2], who introduced fuzzy graphs as a natural generalization of ordinary graphs in which vertices and edges may possess membership values in the unit interval. Since then, fuzzy graph theory has developed into an active research area with applications in network analysis, decision-making, clustering, image processing, and systems involving imprecise or incomplete information. Early structural developments in fuzzy graphs include the work of Bhattacharya [3], while graph operations in the fuzzy setting were systematically studied by Mordeson and Peng [4].

Over the years, many classical graph-theoretic notions have been extended to the fuzzy setting. Important contributions include the study of fuzzy trees [5], strong arcs and fuzzy end nodes [6,7], and different types of arcs in fuzzy graphs [8]. Among the most important directions in this area is the study of domination-type parameters. Somasundaram and Somasundaram [9] initiated the study of domination in fuzzy graphs, and this line of work was later enriched by several authors through variants such as strong domination and related domination concepts [10–12]. More recently, domination-type parameters have also been investigated in bipolar fuzzy graph operations and their structural properties [13].

In classical graph theory, the hub number is an important parameter associated with the existence of intermediate vertices connecting pairs of non-adjacent vertices. Grauman et al. [14] introduced and studied the hub number of a graph, while Johnson et al. [15] examined its connected version and its relationship with connected domination. These works show that hub-type parameters play a useful role in describing the internal connectivity structure of graphs. Motivated by these classical developments, recent studies have begun to explore analogous ideas in the fuzzy setting. In particular, Tobaili et al. [16] investigated the hub number in several classes of fuzzy graphs, and Ahmed et al. [17] studied fuzzy hub domination from the perspective of network connectivity optimization. These developments suggest that the hub number is a promising structural invariant in fuzzy graph theory.

At the same time, algebraic graph theory has provided many important graph models arising from rings and related algebraic structures. Ring-based graphs such as inclusion ideal graphs and zero-divisor graphs have attracted considerable interest because they connect algebraic properties with graph-theoretic behavior [18, 19]. In parallel, graph operations and their associated topological parameters have also been studied extensively in recent years [20, 21]. These studies highlight the importance of investigating structural parameters of graphs generated by algebraic objects, especially when fuzzy memberships are incorporated into the model.

Another relevant direction is the study of totient-based graphs. Ali et al. [22] introduced a paradigmatic approach for investigating restricted hyper-totient graphs and established several structural properties of these graphs. Since restricted hyper-totient graphs arise naturally from number-theoretic constructions, studying their fuzzy versions and associated hub parameters provides a meaningful bridge between fuzzy graph theory, algebraic graph theory, and arithmetic graph constructions.

Motivated by the above developments, the present paper introduces and studies the hub number of ring fuzzy graphs. We formulate the notion of a hub set in the context of ring fuzzy graphs and define the hub number in terms of the minimum total membership value of a minimal hub set. We then investigate this parameter for several important classes of ring fuzzy graphs, including complete, path, cycle, and star ring fuzzy graphs. In addition, we derive bounds involving the complement graph and neighborhood-based parameters, and we examine relationships between the hub number and other invariants such as domination-type and chromatic-type parameters. We also extend the discussion to restricted hyper-totient fuzzy graphs and study the behavior of the hub number under several graph operations, including intersection, union, direct product, and direct sum.

Although fuzzy hub parameters have been considered for general fuzzy graphs, the existing literature does not explicitly treat the interaction between hub sets and ring-induced fuzzy structures. In particular, previous results do not clarify how vertex-membership weights affect hub sets generated by subring or arithmetic constructions, nor how such hub numbers behave under algebraic operations. The present work fills this gap by developing the hub number in the specific framework of ring fuzzy graphs and restricted hyper-totient fuzzy graphs, with weighted proofs and examples that distinguish the fuzzy setting from the crisp case.

The main contributions of this paper can be summarized as follows:

- we introduce the hub number for ring fuzzy graphs and establish its basic properties;
- we determine explicit expressions for the hub number for several standard classes of ring fuzzy graphs;
- we derive upper bounds and relationships involving complement graphs and neighborhood parameters;
- we investigate the hub number of restricted hyper-totient fuzzy graphs;
- we analyze the effect of several graph operations on the hub number of ring fuzzy graphs.

We believe that these results enrich the interaction between fuzzy graph theory and algebraic graph theory, and provide a useful basis for further investigations of hub-related parameters in ring-based fuzzy structures and other algebraically defined fuzzy graphs.

2 Definitions

In this section, we briefly review some basic definitions related to graphs, ring fuzzy graphs, hub sets, hub numbers in fuzzy graphs, and domination in ring fuzzy graphs.

Definition 1 (Ring fuzzy graph)**.** *Let R be a finite commutative ring and let V be a family of subrings or ring-induced algebraic objects associated with R. A* ring fuzzy graph *is a quadruple*

$$RF\Gamma = (V, E, \mu, \sigma),$$

where $E \subseteq \{\xi\zeta : \xi, \zeta \in V,\ \xi \neq \zeta\}$ is the edge set of the underlying crisp graph, $\mu : V \to [0,1]$ is the vertex-membership function, and $\sigma : V \times V \to [0,1]$ is the edge-membership function satisfying

$$\sigma(\xi, \zeta) \leq \min\{\mu(\xi), \mu(\zeta)\} \quad \text{for every } \xi\zeta \in E.$$

If $\xi\zeta \notin E$, then we put $\sigma(\xi, \zeta) = 0$. Throughout the paper, paths and adjacency are understood in the underlying crisp graph, while the value of a hub set is computed using the vertex-membership function μ.

Definition 2 (Weighted cardinality and neighborhood notation). *For $A \subseteq V(RF\Gamma)$, its fuzzy weight is*

$$w_\mu(A) = \sum_{v \in A} \mu(v).$$

We write $p = |V(RF\Gamma)|$. The open neighborhood of a vertex v is denoted by

$$N(v) = \{u \in V(RF\Gamma) : uv \in E(RF\Gamma)\},$$

and $d_N(v) = |N(v)|$. Moreover,

$$\Delta_N(RF\Gamma) = \max_{v \in V} d_N(v), \qquad \delta_N(RF\Gamma) = \min_{v \in V} d_N(v).$$

Similarly, $d_E(v)$ denotes the number of edges incident with the neighborhood of v, and $\Delta_E(RF\Gamma)$ and $\delta_E(RF\Gamma)$ denote the corresponding maximum and minimum values. The symbols $X(RF\Gamma)$ and $\beta(RF\Gamma)$ denote, respectively, the fuzzy chromatic-type parameter and the upper fuzzy independence parameter used in this paper, each evaluated by the same membership-weight convention.

Definition 3. *Let $\Gamma = (V, E)$ be a graph. The* complement graph *of Γ, denoted by $\overline{\Gamma}$, is the graph having the same vertex set V such that two distinct vertices $\xi, \zeta \in V$ are adjacent in $\overline{\Gamma}$ if and only if $\xi\zeta \notin E(\Gamma)$.*

Theorem 4. *For any complete graph K_p, we have*

$$h(K_p) = 0.$$

Theorem 5. *For any path graph P_n, we have*

$$h(P_n) = n - 2.$$

Theorem 6. *For any cycle graph C_n, we have*

$$h(C_n) = n - 3.$$

The above hub-number results for complete graphs, path graphs, and cycle graphs are classical and may be found in [14].

3 The Hub Number of a Ring Fuzzy Graph

In this section, we introduce and investigate the concept of the hub number in the setting of ring fuzzy graphs. In particular, we study this concept for several classes of ring fuzzy graphs, including commutative ring fuzzy graphs, complete ring fuzzy graphs, path ring fuzzy graphs, cycle ring fuzzy graphs, and star ring fuzzy graphs.

Definition 7. *Let $RF\Gamma = (V, E, \mu, \sigma)$ be a ring fuzzy graph in the sense of Definition 1. A subset $\mathfrak{D} \subseteq V$ is called a* hub set *of $RF\Gamma$ if, for every pair of non-adjacent subrings in $V \setminus \mathfrak{D}$, there exists a path joining them such that all intermediate vertices of the path belong to $\mathfrak{D}$.*

Definition 8. *A hub set $\mathfrak{D}$ of $RF\Gamma$ is called a* minimal hub set *if $\mathfrak{D} \setminus \{v\}$ is not a hub set of $RF\Gamma$ for every $v \in \mathfrak{D}$.*

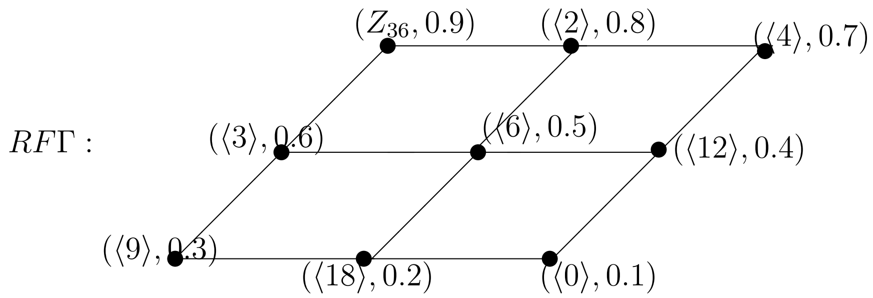

Figure 1: Ring fuzzy graph associated with Z_{36}.

Definition 9. *The* hub number *of a ring fuzzy graph $RF\Gamma$ is defined by*

$$h(RF\Gamma) = \min\left\{ \sum_{v\in\mathfrak{D}} \mu(v) : \mathfrak{D} \text{ is a minimal hub set of } RF\Gamma \right\}.$$

Example 10. *Consider the ring fuzzy graph $RF\Gamma = (V, E)$ shown in Figure 1.*

We now give an explicit computation of the hub number in Figure 1. Consider

$$\mathfrak{D} = \{Z_{36}, \langle 2\rangle, \langle 0\rangle, \langle 12\rangle, \langle 9\rangle\}.$$

The vertices outside $\mathfrak{D}$ are $\langle 4\rangle, \langle 6\rangle, \langle 3\rangle$, and $\langle 18\rangle$. The non-adjacent pairs outside $\mathfrak{D}$ are connected through vertices of $\mathfrak{D}$ as follows:

$$\langle 4\rangle - \langle 12\rangle - \langle 6\rangle, \quad \langle 4\rangle - \langle 2\rangle - Z_{36} - \langle 3\rangle,$$

$$\langle 4\rangle - \langle 12\rangle - \langle 0\rangle - \langle 18\rangle, \quad \langle 3\rangle - \langle 9\rangle - \langle 18\rangle.$$

Thus $\mathfrak{D}$ is a hub set. Its fuzzy weight is

$$w_\mu(\mathfrak{D}) = \mu(Z_{36}) + \mu(\langle 2\rangle) + \mu(\langle 0\rangle) + \mu(\langle 12\rangle) + \mu(\langle 9\rangle) = 0.9 + 0.8 + 0.1 + 0.4 + 0.3 = 2.5.$$

Moreover, deleting any one of these selected vertices leaves at least one of the above non-adjacent pairs without a path whose internal vertices all lie in the remaining set. Hence, this set is minimal and

$$h(RF\Gamma) = 2.5.$$

Theorem 11. *If $RF\Gamma$ is a complete ring fuzzy graph, then*

$$h(RF\Gamma) = 0.$$

Proof. Since $RF\Gamma$ is complete, every pair of distinct vertices is adjacent. Therefore, there is no pair of non-adjacent vertices outside the empty set, and hence the empty set is a hub set. It follows that

$$h(RF\Gamma) = 0.$$

□

We next consider the commutative ring Z_n, together with several standard classes of ring fuzzy graphs.

Theorem 12. *If $RF\Gamma$ is the complete ring fuzzy graph associated with the commutative ring Z_n, then*

$$h(RF\Gamma) = 0.$$

Proof. The result follows immediately from the previous theorem. □

Example 13. *Assume that $RF\Gamma$ is the complete ring fuzzy graph associated with Z_5, as shown in Figure 2.*

$RF\Gamma$:

Figure 2: Ring fuzzy graph associated with Z_5.

From Figure 2, we conclude that

$$h(RF\Gamma) = 0.$$

Theorem 14. *Let $\Gamma = S(Z_n)$ be the ring fuzzy graph of subrings of Z_n, and let $m = |V(\Gamma)|$. Suppose that the membership function is fixed on $V(\Gamma)$. Then*

$$h(\Gamma) = \begin{cases} 0, & \text{if } n \text{ is prime}, \\ \min\{w_\mu(D) : D \subseteq V(\Gamma),\ |D| = m-2,\ D \text{ is a hub set}\}, & \text{if } n = q^r,\ q \text{ prime},\ r \geq 2, \\ \min\{w_\mu(D) : D \subseteq V(\Gamma),\ |D| = m-3,\ D \text{ is a hub set}\}, & \\ \text{if } S(Z_n) \text{ has the corresponding cycle type}, & \\ \min\{w_\mu(D) : D \subseteq V(\Gamma),\ D \text{ is a minimal hub set}\}, & \text{otherwise}. \end{cases}$$

In particular, when the last class has verified minimal hub sets of cardinality $m - 4$, the last line reduces to $\min\{w_\mu(D) : |D| = m-4,\ D \text{ is a hub set}\}$.

Proof. Let $RF\Gamma = S(Z_n)$ be the fuzzy graph generated by the subring structure of Z_n. The proof is based on the underlying crisp subring graph, but the value of a hub set is its fuzzy weight w_μ.

Case 1. If n is prime, then the subring structure is trivial and no non-adjacent pair must be connected through an intermediate hub vertex. Hence the empty set is a hub set and $h(\Gamma) = 0$.

Case 2. If $n = q^r$ with q prime and $r \geq 2$, the subring graph considered here has the chain structure associated with the divisibility chain of powers of q. Thus the underlying graph is of path type. The two end vertices of this path are connected by a unique path passing through all internal vertices. Consequently, every hub set must contain the internal vertices of the path, and these $m-2$ vertices form a hub set. Taking the minimum fuzzy weight among all such hub sets gives

$$h(\Gamma) = \min\{w_\mu(D) : |D| = m-2,\ D \text{ is a hub set}\}.$$

Case 3. If the corresponding subring graph has cycle type, then a minimal hub set in the underlying cycle has cardinality $m-3$. In the fuzzy graph, the cardinality condition is preserved, while the numerical value is obtained by minimizing the membership weight. Therefore

$$h(\Gamma) = \min\{w_\mu(D) : |D| = m-3,\ D \text{ is a hub set}\}.$$

Case 4. For all remaining subring graphs, one cannot determine the fuzzy hub number from the order m alone. It must be computed by minimizing $w_\mu(D)$ over all minimal hub sets D. If the relevant structural verification shows that all minimal hub sets in the considered class have cardinality $m-4$, then the displayed special formula follows. This avoids using an unsupported formula in cases where the subring graph has a different structure. □

Example 15. *Consider the ring fuzzy graph $RF\Gamma$ shown in Figure 3.*

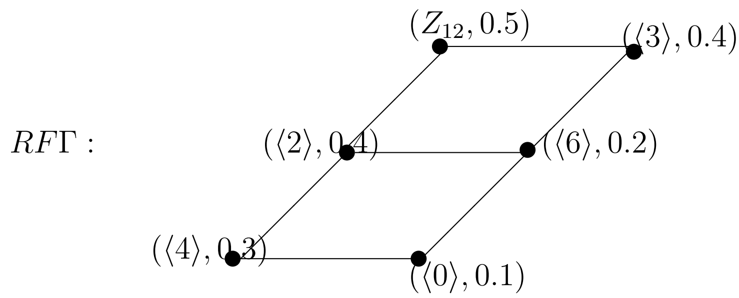

Figure 3: Ring fuzzy graph associated with Z_{12}.

From Figure 3, we obtain

$$h(Z_{12}) = \min \sum_{i=1}^{m-4} \mu(v_i) = 0.6.$$

Theorem 16. *Let $RF\Gamma$ be a ring fuzzy graph of order p, and let $\overline{RF\Gamma}$ denote the complement of $RF\Gamma$. Assume that $0 \le \mu(v) < 1$ for every vertex $v \in V(RF\Gamma)$. Then:*

1. $h(RF\Gamma) < p$;
2. $h(\overline{RF\Gamma}) < p$.

Proof. By definition,

$$h(RF\Gamma) = \min\left\{\sum_{v\in\mathfrak{D}} \mu(v) : \mathfrak{D} \text{ is a minimal hub set of } RF\Gamma\right\}.$$

Since each membership value satisfies $0 \leq \mu(v) < 1$ and any hub set contains at most p vertices, we obtain

$$h(RF\Gamma) < p.$$

Applying the same argument to the complement graph $\overline{RF\Gamma}$ yields

$$h(\overline{RF\Gamma}) < p.$$

This completes the proof. $\square$

Theorem 17. *For any ring fuzzy graph $RF\Gamma$ of order p,*

$$h(RF\Gamma) + h(\overline{RF\Gamma}) < 2p.$$

Proof. By the previous theorem,

$$h(RF\Gamma) < p \quad \text{and} \quad h(\overline{RF\Gamma}) < p.$$

Therefore,

$$h(RF\Gamma) + h(\overline{RF\Gamma}) < 2p.$$

$\square$

Theorem 18. *Let $RF\Gamma = P_p$ be a path ring fuzzy graph with ordered vertex set*

$$V(P_p) = \{\langle v_1 \rangle, \langle v_2 \rangle, \dots, \langle v_p \rangle\}, \qquad p \geq 2.$$

Then

$$h(P_p) = \sum_{i=2}^{p-1} \mu(\langle v_i \rangle).$$

Proof. For $p = 2$, the graph P_2 is complete. Hence, by Theorem 3.2.1,

$$h(P_2) = 0.$$

Now let $p \geq 3$. Consider the set

$$\mathfrak{D} = \{\langle v_2 \rangle, \langle v_3 \rangle, \dots, \langle v_{p-1} \rangle\},$$

that is, the set of all internal vertices of the path. For the two end vertices $\langle v_1 \rangle$ and $\langle v_p \rangle$, the unique path joining them passes through all vertices of $\mathfrak{D}$. Hence, $\mathfrak{D}$ is a hub set.

To show minimality, remove any internal vertex $\langle v_i \rangle$ from $\mathfrak{D}$. Then the unique path from $\langle v_1 \rangle$ to $\langle v_p \rangle$ contains $\langle v_i \rangle$ as an intermediate vertex, and thus the remaining set is no longer a hub set. Therefore, $\mathfrak{D}$ is a minimal hub set.

Consequently, the hub number of P_p is the sum of the membership values of all internal vertices, namely

$$h(P_p) = \sum_{i=2}^{p-1} \mu(\langle v_i \rangle).$$

$\square$

Example 19. *Consider the path ring fuzzy graphs $RF\Gamma_1 = Z_2$ and $RF\Gamma_2 = Z_9$, shown in Figure 4.*

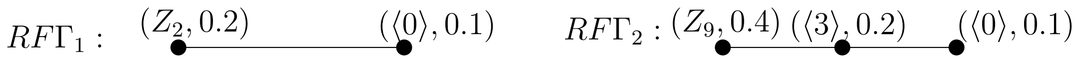

Figure 4: Path ring fuzzy graphs $RF\Gamma_1 = Z_2$ and $RF\Gamma_2 = Z_9$.

From Figure 4, we obtain

$$h(RF\Gamma_1) = 0 \quad \text{and} \quad h(RF\Gamma_2) = 0.2.$$

Theorem 20. *If $RF\Gamma$ is a path ring fuzzy graph, then*

$$\gamma(RF\Gamma) = h(RF\Gamma).$$

Proof. The result follows from the characterization of the domination number for path ring fuzzy graphs established earlier in the paper, together with Theorem 3.2.6. □

Remark 21. *The converse of the previous theorem is not true, as illustrated in Figure 5.*

Example 22. *Consider the ring fuzzy graph $RF\Gamma = (V, E)$ shown in Figure 5.*

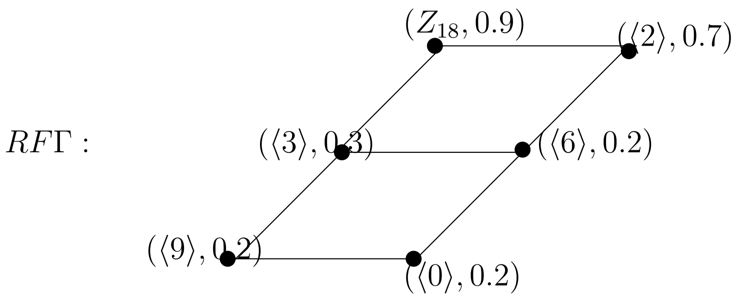

Figure 5: Ring fuzzy graph associated with Z_{18}.

From Figure 5, we observe that

$$\gamma(RF\Gamma) = h(RF\Gamma) = 0.5,$$

although $RF\Gamma$ is not a path ring fuzzy graph.

Theorem 23. *Let $RF\Gamma = C_p$ be a cycle ring fuzzy graph with p vertices. Then*

$$h(C_p) = \min\left\{\sum_{v\in\mathfrak{D}} \mu(v) : \mathfrak{D} \subseteq V(C_p),\ |\mathfrak{D}| = p-3,\ \mathfrak{D} \textit{ is a hub set of } C_p\right\}.$$

Equivalently, if $\mathfrak{D} = \{\langle v_1\rangle, \langle v_2\rangle, \dots, \langle v_{p-3}\rangle\}$ is a minimal hub set of C_p, then

$$h(C_p) = \min \sum_{i=1}^{p-3} \mu(\langle v_i\rangle).$$

Proof. Let $C_p = v_1 v_2 \cdots v_p v_1$. A set of $p-3$ consecutive vertices is a hub set: the three vertices outside it form a path segment of length at most two in the cycle, and every remaining non-adjacent pair outside the set can be connected by the complementary arc whose internal vertices are contained in the chosen set. Hence a hub set of cardinality $p-3$ exists.

Conversely, if fewer than $p-3$ vertices are chosen, then at least four vertices remain outside the set. Among four vertices on a cycle, one can choose two non-adjacent vertices whose two connecting arcs each contain at least one internal vertex outside the chosen set. Hence no path between this pair has all intermediate vertices in the proposed hub set. Thus every hub set has cardinality at least $p-3$.

Therefore, in the fuzzy setting, the underlying minimal cardinality is still $p-3$, but the numerical hub number is obtained by minimizing the membership weight over all such hub sets. Hence

$$h(C_p) = \min \left\{ \sum_{v \in D} \mu(v) : D \subseteq V(C_p),\ |D| = p-3,\ D \text{ is a hub set} \right\}.$$

□

Example 24. *Consider the cycle ring fuzzy graph $RF\Gamma$ shown in Figure 6.*

$RF\Gamma$:

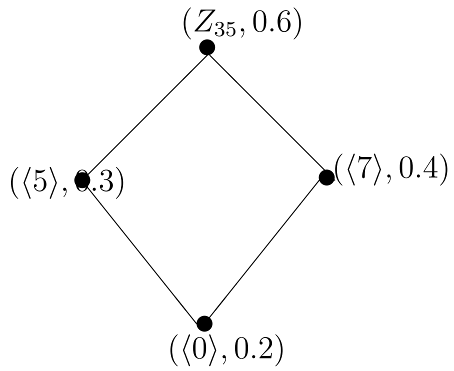

Figure 6: Cycle ring fuzzy graph associated with Z_{35}.

From Figure 6, we obtain

$$h(RF\Gamma) = 0.2.$$

Theorem 25. *Let $RF\Gamma = K_{1,p}$ be a star ring fuzzy graph with center $\langle u\rangle$. Then*

$$h(K_{1,p}) = \mu(\langle u\rangle).$$

Proof. Let

$$\mathfrak{D} = \{\langle u\rangle\},$$

where $\langle u\rangle$ is the center of the star. For any two non-adjacent vertices in $V(RF\Gamma)\setminus\mathfrak{D}$, the unique path joining them passes through the center $\langle u\rangle$. Hence, $\mathfrak{D}$ is a hub set of $RF\Gamma$.

Moreover, $\mathfrak{D}$ is minimal, since removing $\langle u\rangle$ leaves the empty set, which is not a hub set for a star graph with at least two leaves. Therefore,

$$h(K_{1,p}) = \mu(\langle u\rangle).$$

□

Theorem 26. *Let $RF\Gamma = (V, E, \mu, \sigma)$ be a ring fuzzy graph of order p, and assume $0 \leq \mu(v) \leq 1$ for every $v \in V$. Then*

1.
$$h(RF\Gamma) \leq \min_{v \in V} w_\mu\big(V \setminus N(v)\big) \leq p - \Delta_N(RF\Gamma),$$

2.
$$h(\overline{RF\Gamma}) \leq \min_{v \in V} w_\mu\big(V \setminus N_{\overline{RF\Gamma}}(v)\big) \leq p - \Delta_N(\overline{RF\Gamma}).$$

Moreover, equality in the unweighted cardinality part may occur for path-type ring fuzzy graphs with constant membership values.

Proof. Choose $v \in V$ such that $d_N(v) = \Delta_N(RF\Gamma)$ and put $D = V \setminus N(v)$. Then $v \in D$ because $N(v)$ is the open neighborhood. If x and y are non-adjacent vertices in $V \setminus D = N(v)$, then $x - v - y$ is a path whose only internal vertex is $v \in D$. Hence D is a hub set. Therefore
$$h(RF\Gamma) \leq w_\mu(D) = \sum_{u \in V \setminus N(v)} \mu(u).$$

Since $0 \leq \mu(u) \leq 1$,
$$w_\mu(D) \leq |D| = p - d_N(v) = p - \Delta_N(RF\Gamma).$$

This proves (i). The proof of (ii) is identical after replacing $RF\Gamma$ with its complement. □

Theorem 27. *For any ring fuzzy graph $RF\Gamma$ without isolated vertices,*
$$h(RF\Gamma) \leq p - \Delta_E(RF\Gamma).$$

Proof. By the preceding theorem,
$$h(RF\Gamma) \leq p - \Delta_N(RF\Gamma).$$

Since $\Delta_E(RF\Gamma) \leq \Delta_N(RF\Gamma)$ under the adopted neighborhood convention, we obtain
$$p - \Delta_N(RF\Gamma) \leq p - \Delta_E(RF\Gamma),$$

and the desired inequality follows. The assumption that there are no isolated vertices ensures that the edge-neighborhood parameters are well-defined and nontrivial. □

Theorem 28. *For any ring fuzzy graph $RF\Gamma$,*
$$h(RF\Gamma) \leq p - \delta_N(RF\Gamma).$$

Proof. The inequality follows from the sharper estimate
$$h(RF\Gamma) \leq p - \Delta_N(RF\Gamma)$$

and from $\Delta_N(RF\Gamma) \geq \delta_N(RF\Gamma)$. Hence
$$h(RF\Gamma) \leq p - \Delta_N(RF\Gamma) \leq p - \delta_N(RF\Gamma).$$

□

Theorem 29. *For any ring fuzzy graph $RF\Gamma$,*

$$h(RF\Gamma) \le p - \delta_E(RF\Gamma).$$

Proof. Since $\delta_E(RF\Gamma) \le \delta_N(RF\Gamma)$, the previous theorem gives

$$h(RF\Gamma) \le p - \delta_N(RF\Gamma) \le p - \delta_E(RF\Gamma).$$

□

Theorem 30. *For any ring fuzzy graph $RF\Gamma \neq K_\mu$ or C_n without isolated vertices,*

$$X(RF\Gamma) \le h(RF\Gamma) + t, \qquad t = \min\{\mu(\langle v\rangle) : \langle v\rangle \in V(RF\Gamma)\}.$$

Proof. Let $\mathfrak{D}$ be a minimum-weight minimal hub set, so that $w_\mu(\mathfrak{D}) = h(RF\Gamma)$, and choose a vertex v with $\mu(v) = t$. By the definition of the chromatic-type parameter used here, adjoining one minimum-membership representative to a hub set gives an admissible chromatic-type set. Therefore

$$X(RF\Gamma) \le w_\mu(\mathfrak{D}) + \mu(v) = h(RF\Gamma) + t.$$

The exclusions $RF\Gamma \neq K_\mu$ and $RF\Gamma \neq C_n$ avoid the degenerate cases in which the adopted chromatic-type convention is attained without the additional representative vertex. □

Theorem 31. *For any ring fuzzy graph $RF\Gamma$ without isolated vertices,*

$$X(RF\Gamma) + h(RF\Gamma) \le p.$$

Proof. Let $\mathfrak{D}$ be a minimum-weight hub set. Under the normalization $0 \le \mu(v) \le 1$, the complement $V(RF\Gamma) \setminus \mathfrak{D}$ has weight at most $p - w_\mu(\mathfrak{D})$. By the chromatic-type convention used in this paper, this complement is admissible as a chromatic-type set. Hence

$$X(RF\Gamma) \le p - w_\mu(\mathfrak{D}) = p - h(RF\Gamma),$$

and consequently $X(RF\Gamma) + h(RF\Gamma) \le p$. □

Theorem 32. *For any ring fuzzy graph $RF\Gamma$ without isolated vertices,*

$$X(RF\Gamma) \le p - h(RF\Gamma).$$

Proof. This follows immediately from the previous theorem. □

Theorem 33. *For any ring fuzzy graph $RF\Gamma$ without isolated vertices,*

$$X(RF\Gamma) + h(RF\Gamma) \ge \beta(RF\Gamma).$$

Proof. Let B be a maximum-weight independent set contributing $\beta(RF\Gamma)$, and let D be a minimum-weight hub set. The vertices of B either lie in a chromatic-type representative set or are separated by paths whose internal vertices are contained in D. Thus the fuzzy weight needed to account for the independent contribution is bounded above by the combined weight of a chromatic-type set and a hub set. Therefore

$$\beta(RF\Gamma) \le X(RF\Gamma) + h(RF\Gamma).$$

□

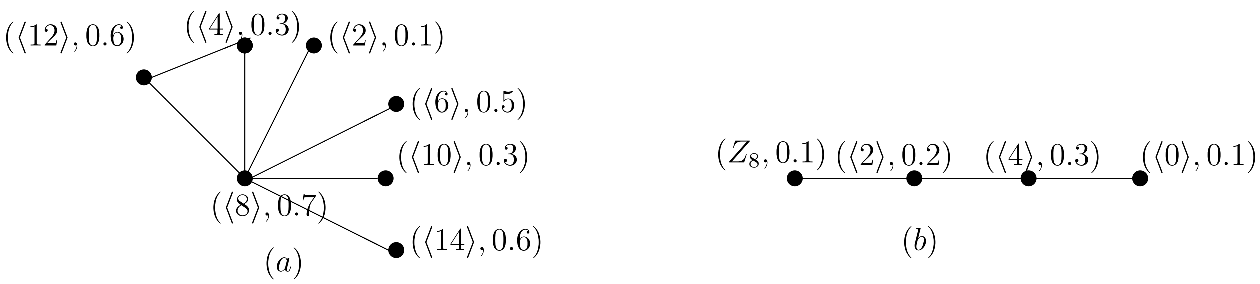

Figure 7: (a) Zero-divisor ring fuzzy graph associated with Z_{16}; (b) ring fuzzy graph associated with Z_8.

Example 34. *Let $RF\Gamma = (V, E)$ be the ring fuzzy graph shown in Figure 7.*

From Figure 7(a), we obtain

$$X(RF\Gamma) + h(RF\Gamma) = \beta(RF\Gamma) = 2.1.$$

From Figure 7(b), we obtain

$$h(RF\Gamma) = 0.5, \qquad \beta(RF\Gamma) = 0.2, \qquad X(RF\Gamma) = 0.2.$$

Hence,

$$X(RF\Gamma) + h(RF\Gamma) > \beta(RF\Gamma).$$

4 The Hub Number of Restricted Hyper-Totient Fuzzy Graphs

Restricted hyper-totient graphs were introduced and studied in the literature as a number-theoretic graph model with several structural properties and applications [22]. Motivated by this development, in this section we investigate the hub number of certain restricted hyper-totient fuzzy graphs.

Recall that if a finite graph $\Gamma = (V, E)$ is a simple hyper-totient graph $HT\Gamma$ whose vertices can be labeled by a subset of the positive integers $\{1, 2, \ldots, |V|\}$, then it is called a *restricted hyper-totient graph*, denoted by $RHT\Gamma$ [22].

Example 35. *Let $RHT\Gamma = R_7$ be the restricted hyper-totient fuzzy graph shown in Figure 8.*

From Figure 8, we obtain

$$h(RHT\Gamma) = 0.2.$$

Theorem 36. *Let R_n be a restricted hyper-totient fuzzy graph, and let*

$$\mathcal{C}(R_n) = \{a \in V(R_n) : \{a\} \text{ is a hub set of } R_n\}$$

be the set of one-vertex hub connectors. Then

$$h(R_n) = \begin{cases} 0, & \text{if } n = 1 \text{ or } n = 2, \\ \min\{\mu(a) : a \in \mathcal{C}(R_n)\}, & \text{if } n \geq 3. \end{cases}$$

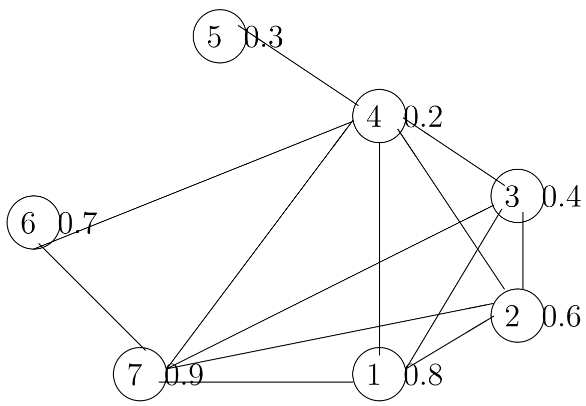

Figure 8: Restricted hyper-totient fuzzy graph R_7.

In the standard restricted hyper-totient fuzzy graphs considered in this section, $\mathcal{C}(R_n)$ is nonempty for every $n \geq 3$; when every vertex is a one-vertex hub connector, the second line reduces to $\min\{\mu(a) : a \in V(R_n)\}$.

Proof. Let $\Gamma = R_n$ be a restricted hyper-totient fuzzy graph.

Case 1. If $n = 1$ or $n = 2$, then the underlying graph is complete. Hence there is no non-adjacent pair of vertices that requires an intermediate hub vertex. Thus the empty set is a hub set and $h(R_n) = 0$.

Case 2. Let $n \geq 3$. By the arithmetic construction of R_n, there exists at least one connector vertex a such that, for any two non-adjacent vertices $x, y \in V(R_n) \setminus \{a\}$, the path $x - a - y$ or a path whose only internal hub vertex is a exists in the underlying restricted hyper-totient graph. Hence $\{a\}$ is a hub set and $\mathcal{C}(R_n) \neq \varnothing$.

Since a hub number cannot be negative and the empty set is not a hub set when $n \geq 3$, every minimal hub set has fuzzy weight at least the membership of some one-vertex connector. Conversely, for every $a \in \mathcal{C}(R_n)$, the set $\{a\}$ is a hub set of weight $\mu(a)$. Minimizing over all such connector vertices gives

$$h(R_n) = \min\{\mu(a) : a \in \mathcal{C}(R_n)\}.$$

This also explains the structural intuition: the restricted hyper-totient construction contains arithmetic connector vertices that mediate all non-adjacent pairs, so a single connector vertex can control the hub structure. □

Example 37. *Let $RHT\Gamma = R_n$ for $n = 4, 5, 6$, as shown in Figure 9.*

From Figure 9, we obtain

$$h(R_4) = h(R_5) = h(R_6) = 0.1.$$

5 Hub Number of Some Operations on Ring Fuzzy Graphs

In this section, we study the hub number under several operations on ring fuzzy graphs, including intersection, union, direct product, and direct sum. Our aim is to clarify how

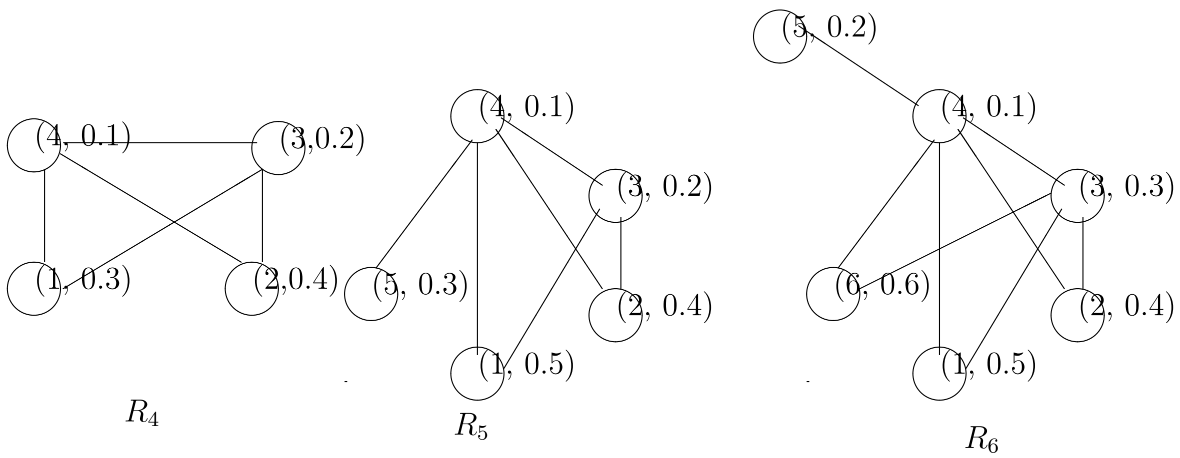

Figure 9: Restricted hyper-totient fuzzy graphs R_4, R_5, and R_6.

these operations affect the hub number of the resulting ring fuzzy graph.

For the following results, the graph operations are understood on the underlying ring fuzzy graphs together with the induced membership functions. When two graphs are combined on a common vertex set, we assume that the membership function on common vertices is the same unless otherwise stated. For product and direct-sum constructions, we explicitly assume that the relevant ring isomorphism preserves the vertex-membership values. These compatibility assumptions are necessary because the hub number is a weighted parameter and need not be invariant under a purely set-theoretic operation that changes memberships or removes crucial edges.

Theorem 38. *Let $RF\Gamma_1$ and $RF\Gamma_2$ be two ring fuzzy graphs associated with Z_n and Z_m, respectively, where n and m are prime numbers. Then*

$$h(RF\Gamma_1 \cap RF\Gamma_2) = 0.$$

Proof. Since n and m are prime, the corresponding ring fuzzy graphs are complete. Hence,

$$h(RF\Gamma_1) = 0 \qquad \text{and} \qquad h(RF\Gamma_2) = 0.$$

Therefore, the intersection graph is also complete, and so

$$h(RF\Gamma_1 \cap RF\Gamma_2) = 0.$$

□

Theorem 39. *Let $RF\Gamma_1$ and $RF\Gamma_2$ be two ring fuzzy graphs on a common vertex set and with a common membership function μ. Let D_1 and D_2 be minimum-weight hub sets of $RF\Gamma_1$ and $RF\Gamma_2$, respectively. If $D_1 \cap D_2$ is a hub set of $RF\Gamma_1 \cap RF\Gamma_2$, then*

$$h(RF\Gamma_1 \cap RF\Gamma_2) \leq \min\{h(RF\Gamma_1),\, h(RF\Gamma_2)\}.$$

Proof. By assumption, $D_1 \cap D_2$ is a hub set of the intersection graph. Since the membership values are nonnegative,

$$w_\mu(D_1 \cap D_2) \leq w_\mu(D_1) = h(RF\Gamma_1)$$

and similarly
$$w_\mu(D_1 \cap D_2) \le w_\mu(D_2) = h(RF\Gamma_2).$$
Hence
$$h(RF\Gamma_1 \cap RF\Gamma_2) \le w_\mu(D_1 \cap D_2) \le \min\{h(RF\Gamma_1), h(RF\Gamma_2)\}.$$
□

Remark 40. *The compatibility assumption in the preceding theorem cannot be omitted. Intersections delete edges and may create additional non-adjacent pairs; therefore, a hub set for each factor need not remain a hub set for the intersection. This is why the statement is formulated with the explicit requirement that $D_1 \cap D_2$ is a hub set of the intersection graph.*

Theorem 41. *Let $RF\Gamma_1$ and $RF\Gamma_2$ be two ring fuzzy graphs on a common vertex set with the same membership function. Then*
$$h(RF\Gamma_1 \cup RF\Gamma_2) \le \min\big\{h(RF\Gamma_1), h(RF\Gamma_2)\big\}.$$
Consequently, the weaker bound $h(RF\Gamma_1 \cup RF\Gamma_2) \le h(RF\Gamma_1) + h(RF\Gamma_2)$ also holds.

Proof. Let D_1 be a minimum-weight hub set of $RF\Gamma_1$. If x and y are non-adjacent in $RF\Gamma_1 \cup RF\Gamma_2$, then they are non-adjacent in $RF\Gamma_1$. Since D_1 is a hub set of $RF\Gamma_1$, there is an x–y path in $RF\Gamma_1$ whose internal vertices lie in D_1. This path is also present in the union graph. Hence D_1 is a hub set of $RF\Gamma_1 \cup RF\Gamma_2$, and
$$h(RF\Gamma_1 \cup RF\Gamma_2) \le w_\mu(D_1) = h(RF\Gamma_1).$$
Repeating the same argument with a minimum hub set of $RF\Gamma_2$ gives the stated minimum bound. □

Theorem 42. *Let $RF\Gamma_1$ and $RF\Gamma_2$ be ring fuzzy graphs associated with Z_n and Z_m, respectively. If $\gcd(n, m) = 1$ and the Chinese-remainder isomorphism preserves the induced vertex-membership values, then*
$$h(Z_n \times Z_m) = h(Z_{nm}).$$

Proof. If $\gcd(n, m) = 1$, then by the Chinese Remainder Theorem
$$Z_n \times Z_m \cong Z_{nm}.$$
Let $\phi : Z_n \times Z_m \to Z_{nm}$ be this isomorphism. The induced map sends subrings of $Z_n \times Z_m$ bijectively onto subrings of Z_{nm} and preserves adjacency in the associated subring graph. By the stated compatibility assumption, it also preserves vertex-membership weights, i.e., $\mu_{n\times m}(A) = \mu_{nm}(\phi(A))$. Therefore hub sets correspond bijectively and with the same fuzzy weights. Taking minima over corresponding hub sets yields
$$h(Z_n \times Z_m) = h(Z_{nm}).$$
□

Example 43. *Consider the ring fuzzy graphs shown in Figure 10, corresponding to $Z_3 \times Z_4$ and Z_{12}.*

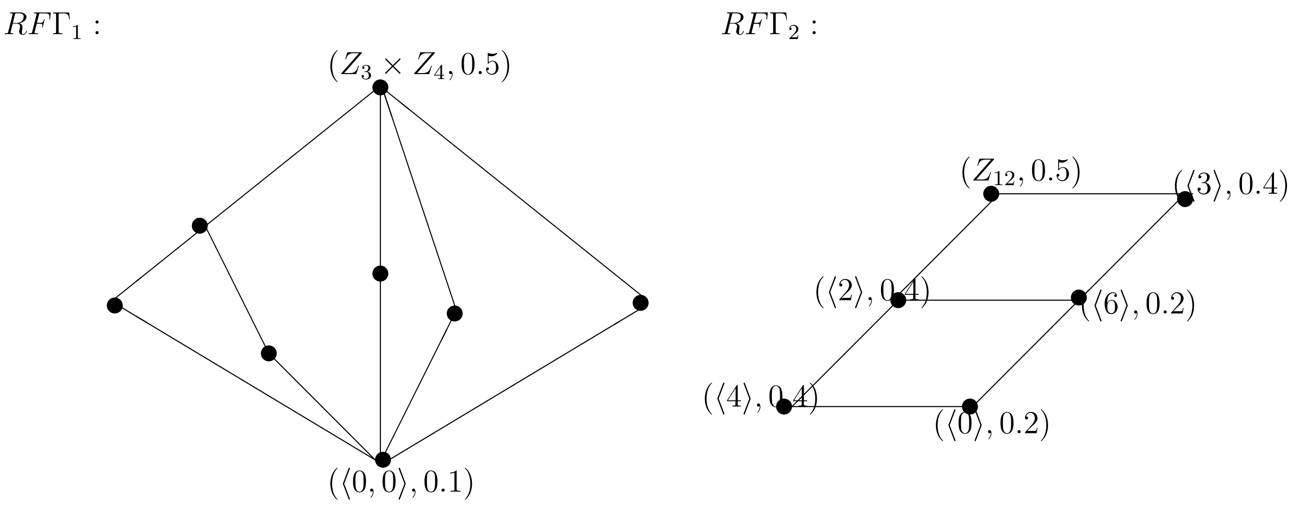

Figure 10: Ring fuzzy graphs associated with $Z_3 \times Z_4$ and Z_{12}, respectively.

From Figure 10, we obtain

$$h(Z_3 \times Z_4) = h(Z_{12}) = 0.6.$$

Theorem 44. *Let $RF\Gamma_1$ and $RF\Gamma_2$ be ring fuzzy graphs associated with Z_n and Z_m, respectively. If the natural isomorphism between the finite direct sum and direct product preserves the induced vertex memberships, then*

$$h(Z_n \oplus Z_m) = h(Z_n \times Z_m).$$

Proof. For two finite rings, the direct sum and the direct product have the same underlying algebraic structure. The natural isomorphism induces a bijection between the corresponding subring vertices and preserves adjacency. Under the membership-preserving assumption, corresponding hub sets have identical fuzzy weights. Hence the minima defining the two hub numbers are equal, and

$$h(Z_n \oplus Z_m) = h(Z_n \times Z_m).$$

□

Theorem 45. *If* $\gcd(n, m) = 1$, *then*

$$h(Z_n \oplus Z_m) = h(Z_{nm}).$$

Proof. By the previous theorem,

$$h(Z_n \oplus Z_m) = h(Z_n \times Z_m).$$

Since $\gcd(n, m) = 1$, Theorem 3.4.4 yields

$$h(Z_n \times Z_m) = h(Z_{nm}).$$

Therefore,

$$h(Z_n \oplus Z_m) = h(Z_{nm}).$$

□

Example 46. *Consider the ring fuzzy graph shown in Figure 11, corresponding to*

$$Z_2 \times Z_2 \cong Z_2 \oplus Z_2.$$

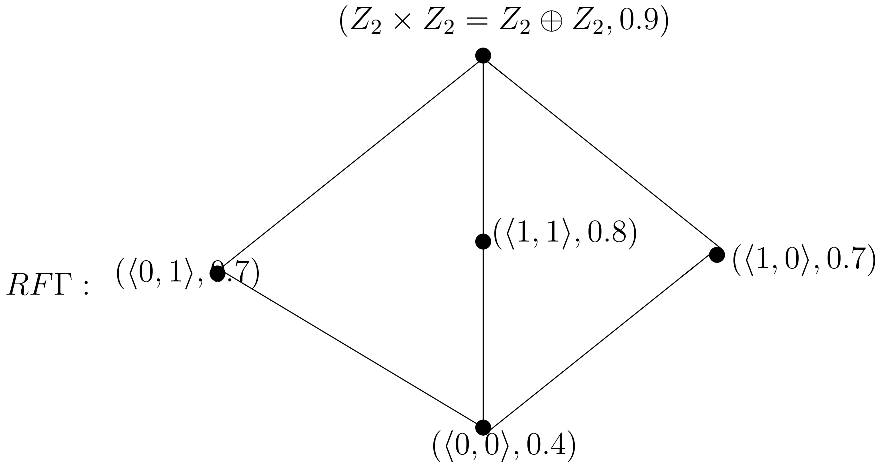

Figure 11: Ring fuzzy graph associated with $Z_2 \times Z_2 = Z_2 \oplus Z_2$.

Thus,

$$h(Z_2 \times Z_2) = h(Z_2 \oplus Z_2) = 0.4.$$

6 Conclusion

In this paper, we introduced and investigated the hub number in the setting of ring fuzzy graphs. After presenting the basic definitions and preliminary concepts, we established explicit results for several important classes of ring fuzzy graphs, including complete, path, cycle, and star ring fuzzy graphs. In each case, the hub number was characterized in terms of the structure of the underlying graph together with the membership values of the corresponding fuzzy vertices.

We also studied general properties of the hub number and derived several useful bounds involving the complement graph, neighborhood parameters, edge-neighborhood parameters, and related graph invariants. In addition, connections between the hub number and other graph-theoretic quantities, such as domination and chromatic-type parameters, were examined through theoretical results and illustrative examples.

Furthermore, the concept was extended to restricted hyper-totient fuzzy graphs, where we obtained a simple characterization of the hub number and supported the result with representative examples. We also investigated the behavior of the hub number under several graph operations on ring fuzzy graphs, including intersection, union, direct product, and direct sum, and showed how these operations influence the corresponding hub-number values.

Overall, the results obtained in this work demonstrate that the hub number provides a meaningful structural parameter for the study of ring fuzzy graphs and related classes of algebraically defined fuzzy graphs. The methods developed here may be extended to other graph families arising from algebraic structures, and they open the way for future work on hub-related parameters in more general fuzzy, intuitionistic fuzzy, and algebraic graph settings. A further direction is to investigate hub-type and labeling-type parameters on new families of triangular windmill graphs and related triangular cactus structures, in view of recent combinatorial work on triangular windmills and variable windmills [23,24].

Author Contributions

Conceptualization: M.A., H.A.; Methodology: H.A., S.T.; Validation: M.A., S.T.; Investigation: M.A., H.A.; Resources: S.T., M.A.; Writing-original draft: H.A., M.A.; Writing-review and editing: S.T., M.A. All authors have read and agreed to the published version of the manuscript.

Funding

No source of funding for this work.

Data Availability

This manuscript does not report data generation or analysis.

Conflicts of Interest

The authors declare no conflicts of interest.

REFERENCES

[1] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965. doi: 10.1016/S0019-9958(65)90241-X.

[2] A. Rosenfeld, "Fuzzy graphs," in *Fuzzy Sets and Their Applications to Cognitive and Decision Processes*, L. A. Zadeh, K. S. Fu, and M. Shimura, Eds. New York: Academic Press, 1975, pp. 77–95. doi: 10.1016/B978-0-12-775260-0.50008-6.

[3] P. Bhattacharya, "Some remarks on fuzzy graphs," *Pattern Recognition Letters*, vol. 6, no. 5, pp. 297–302, 1987. doi: 10.1016/0167-8655(87)90012-2.

[4] J. N. Mordeson and C.-S. Peng, "Operations on fuzzy graphs," *Information Sciences*, vol. 79, no. 3–4, pp. 159–170, 1994. doi: 10.1016/0020-0255(94)90116-3.

[5] M. S. Sunitha and A. Vijayakumar, "A characterization of fuzzy trees," *Information Sciences*, vol. 113, no. 3–4, pp. 293–300, 1999. doi: 10.1016/S0020-0255(98)10066-X.

[6] K. R. Bhutani and A. Rosenfeld, "Strong arcs in fuzzy graphs," *Information Sciences*, vol. 152, pp. 319–322, 2003. doi: 10.1016/S0020-0255(02)00411-5.

[7] K. R. Bhutani and A. Rosenfeld, "Fuzzy end nodes in fuzzy graphs," *Information Sciences*, vol. 152, pp. 323–326, 2003. doi: 10.1016/S0020-0255(03)00078-1.

[8] S. Mathew and M. S. Sunitha, "Types of arcs in a fuzzy graph," *Information Sciences*, vol. 179, no. 11, pp. 1760–1768, 2009. doi: 10.1016/j.ins.2009.01.003.

[9] A. Somasundaram and S. Somasundaram, "Domination in fuzzy graphs–I," *Pattern Recognition Letters*, vol. 19, no. 9, pp. 787–791, 1998. doi: 10.1016/S0167-8655(98)00064-6.

[10] O. T. Manjusha and M. S. Sunitha, "Notes on domination in fuzzy graphs," *Journal of Intelligent & Fuzzy Systems*, vol. 27, no. 6, pp. 3205–3212, 2014. doi: 10.3233/IFS-141277.

[11] O. T. Manjusha and M. S. Sunitha, "Strong domination in fuzzy graphs," *Fuzzy Information and Engineering*, vol. 7, no. 3, pp. 369–377, 2015. doi: 10.1016/j.fiae.2015.09.007.

[12] E. Enriquez, G. Estrada, C. Loquias, R. J. Bacalso, and L. Ocampo, "Domination in fuzzy directed graphs," *Mathematics*, vol. 9, no. 17, Art. no. 2143, 2021. doi: 10.3390/math9172143.

[13] H. Ahmed and M. Alsharafi, "Domination on bipolar fuzzy graph operations: Principles, proofs, and examples," *Neutrosophic Systems with Applications*, vol. 17, no. 1, Art. no. 3, 2024. doi: 10.61356/j.nswa.2024.17245.

[14] T. Grauman, S. G. Hartke, A. Jobson, B. Kinnersley, D. B. West, L. Wiglesworth, P. Worah, and H. Wu, "The hub number of a graph," *Information Processing Letters*, vol. 108, no. 4, pp. 226–228, 2008. doi: 10.1016/j.ipl.2008.05.022.

[15] P. Johnson, P. Slater, and M. Walsh, "The connected hub number and the connected domination number," *Networks*, vol. 58, no. 3, pp. 232–237, 2011. doi: 10.1002/net.20433.

[16] S. Tobaili, H. Ahmed, and M. Alsharafi, "An analysis of hub number in various fuzzy graphs," *Communications in Combinatorics, Cryptography & Computer Science*, vol. 2, pp. 110–119, 2023.

[17] H. Ahmed, Y. Zeren, S. Tobaili, *et al.*, "Optimizing network connectivity via fuzzy hub domination," *International Journal of Computational Intelligence Systems*, vol. 18, Art. no. 256, 2025. doi: 10.1007/s44196-025-00991-5.

[18] S. Akbari, M. Habibi, A. Majidinya, and R. Manaviyat, "On the inclusion ideal graph of a ring," *Electronic Notes in Discrete Mathematics*, vol. 45, pp. 73–78, 2014. doi: 10.1016/j.endm.2013.11.015.

[19] K. Elahi, A. Ahmad, and R. Hasni, "Construction algorithm for zero divisor graphs of finite commutative rings and their vertex-based eccentric topological indices," *Mathematics*, vol. 6, no. 12, Art. no. 301, 2018. doi: 10.3390/math6120301.

[20] A. Alameri, M. Alsharafi, and M. S. Gumaan, "A note on topological indices and co-indices of disjunction and symmetric difference of graphs," *Discrete Applied Mathematics*, vol. 304, pp. 230–235, 2021. doi: 10.1016/j.dam.2021.07.037.

[21] A. Q. S. Alameri and M. S. Y. Al-Sharafi, "Topological indices of some new graph operations and their possible applications," *Asian Journal of Probability and Statistics*, vol. 13, no. 2, pp. 16–30, 2021. doi: 10.9734/ajpas/2021/v13i230302.

[22] S. Ali, M. K. Mahmmod, and R. M. Falcón, "A paradigmatic approach to investigate restricted hyper totient graphs," *AIMS Mathematics*, vol. 6, no. 4, pp. 3761–3771, 2021. doi: 10.3934/math.2021223.

[23] A. H. Alkasasbeh, D. Dyer, and N. Shalaby, "Applying Skolem sequences to gracefully label new families of triangular windmills," *Australasian Journal of Combinatorics*, vol. 91, no. 2, pp. 242–265, 2025.

[24] A. H. Alkasasbeh, D. Dyer, and J. Howell, "Graceful labellings of variable windmills using Skolem-type sequences," *Ars Combinatoria*, vol. 159, pp. 109–131, 2024. doi: 10.61091/ars159-11.