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Research Article

Caputo fractional operator and Aboodh transform technique of WBK equations: Some comparisons of analytical and numerical solutions with other techniques

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Abstract
Among the most famous equations in theory of shallow water waves are Whitham-Broer-Kaup equations (WBKEs) which are defined as a system of coupled nonlinear partial differential equations that describes the propagation of shallow water waves, incorporating different dispersion relations and diffusion powers. In this paper we use Aboodh transform in residual power series method to get a new technique, called Aboodh residual power series technique (ARPST). This technique and together with Caputo fractional operator provide us some analytical and numerical solutions of WBKEs by calculating the coefficients of the power series. We will use ARPST to get the solutions for WBKEs in two cases, with the perfect initial condition and approximate initial condition. For showing ARPST's capability, efficiency and reliability, we describe graphically numerical results for analytical solutions and compare our solutions with other obtained solutions of WBKEs by using Adomian decomposition Method, the optimal homotopy asymptotic method and the Sumudu decomposition method.

Keywords: Residual power series; Caputo derivative operator; Aboodh transform

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1 Introduction

The theory of partial fractional differential equations (PDEs) became important part in mathematical sciences where PDE is considered a good description for many models, objects movements and fluid flows in some fields such as electrical networks, physics, engineering, groundwater problems, fluid mechanics, medicine and polymer science [1]-[4]. In theory of wave motions and shallow water waves, there are several PDEs that are represented and considered as the most important equations in fluid theory, physics and mathematics. among the most famous equations of these equations are WBKEs which are defined as a system of coupled nonlinear partial differential equations that describes the propagation of shallow water waves, incorporating different dispersion relations and diffusion powers. This system is developed by Whitham, Broer, and Kaup [5, 6, 7] where is used in nonlinear physics and fluid dynamics to model phenomena such as wave behavior in theory of shallow water waves. It has been demonstrated that the non-linear development of shallow water waves in fluid dynamics is accurately described by the coupled WBKEs [8] which is given by, in classical order for the phenomenon,

$$\begin{cases} \partial_{\nu_2}\tau(\nu_1,\nu_2) + \tau(\nu_1,\nu_2)\partial_{\nu_1}\tau(\nu_1,\nu_2) + \partial_{\nu_1}\pi(\nu_1,\nu_2) + \beta\partial^2_{\nu_1}\tau(\nu_1,\nu_2) = 0 \\ \partial_{\nu_2}\pi(\nu_1,\nu_2) + \partial_{\nu_1}\left[\tau(\nu_1,\nu_2)\pi(\nu_1,\nu_2)\right] - \beta\partial^2_{\nu_1}\pi(\nu_1,\nu_2) + \beta'\partial^3_{\nu_1}\tau(\nu_1,\nu_2) = 0 \end{cases} \tag{1}$$

where $\partial_{\nu_2} := \frac{\partial}{\partial \nu_2}$, τ denotes the horizontal velocity of the fluid, π denotes the height velocity of the fluid that both of which differ significantly from their equilibrium states and β, β' represent various diffusion powers. Over the past few decades, the investigation of non-linear PDEs has become a major focus of research. Numerous authors have developed various mathematical methods to analyse approximate solutions of these equations. For instance, Mohyud Din et al. [9] explored the analysis of several integer-order PDEs using homotopy perturbation techniques. In solving the coupled set of Burgers and Brusselator equations, Biazar and Aminikhah [10] employed perturbation techniques. Yuzbasi and Sahin [11] presented some solutions for the one-dimensional parabolic convection-diffusion model problems Yang [12] introduced a new integral method to solve the steady heat-transfer problem. Ahmad et al., [13] introduced perturbation method Caputo-Fabrizio PDE. Kumar [14] introduced a powerful method to solve non-linear singular boundary value problems. Various studies have been introduced to investigate solutions for the non-linear WBKEs (1). Mohyud Din et al. [15] utilized perturbation methods. Xie et al. [16] explored solutions using hyperbolic techniques. Ahmad et al.[17] used both the Adomian decomposition method and He's polynomial to solve WBKEs (1). For mathematical systems and modeling in theory of fractional differential equations, several fractional integrate and derivative operators were introduced studied to describe the various material-hereditary properties of processes. Many efficient techniques and methods have been introduced and investigated in getting approximate solutions of physical and biological FPDEs such as the sine–Gordon expansion method for solving WZ system models [18], Laplace transform method [3], the reproducing kernel Hilbert space method with Fornberg-Whitham type equation [19], homotopy perturbation method [20], fractional Newton method [21], an extended direct algebraic mapping method [22], the expansion method [23], a monotone iterative method with reaction diffusion equation [24], a modified Adams–Bashforth method [25], variational iteration method [26] with two systems, with telegraph equation [27] and with non-local fractional equation [28]. The technique ARPST is one of these techniques which was used in solving some fractional differential equations such as Noor et al., [29] solved one-dimensional nonlinear shock waves by ARPST. Liaqat et al., [30] developed a analytic technique ARPST to solve the CTFPDEs. Edalatpanah and Abdolmaleki, [31] solved the N-Wh-S equation. Liaqat et al., [32] solved the Black–Scholes differential equations. Yasmin and Almuqrin, [33] used ARPST to solve some fractional differential equations.

Consider the following Caputo WBKEs in fractional order of the form:

$$\begin{cases} {}^{c}\mathcal{D}^{\omega}_{\nu_2}\tau(\nu_1,\nu_2)+\tau(\nu_1,\nu_2)\partial_{\nu_1}\tau(\nu_1,\nu_2)+\partial_{\nu_1}\pi(\nu_1,\nu_2)+\beta\partial^2_{\nu_1}\tau(\nu_1,\nu_2)=0 \\ {}^{c}\mathcal{D}^{\omega}_{\nu_2}\pi(\nu_1,\nu_2)+\partial_{\nu_1}\left[\tau(\nu_1,\nu_2)\pi(\nu_1,\nu_2)\right]-\beta\partial^2_{\nu_1}\pi(\nu_1,\nu_2)+\beta'\partial^3_{\nu_1}\tau(\nu_1,\nu_2)=0 \end{cases} \tag{2}$$

where ${}^{c}\mathcal{D}^{\omega}_{\nu_2}$ is the Caputo fractional derivative of order $0<\omega\leq 1$. Some techniques and methods have been used in solving the Caputo WBKEs (2) such as Adomian decomposition Method [34], the OHAM [35], the Sumudu decomposition method [36], Yang transformation coupled with the Adomian technique [37], q-Homotopy Analysis Transform Method [38], natural decomposition method [38], the scaling transform method[39], redefined quintic B-spline basis and Von Neumann technique [40], the residual power series method[41] and Laplace transform coupled with Adomian decomposition method [42].

In this paper, we investigate and introduce the efficiency and analytical scheme of ARPSM by calculating analytical solutions of Caputo WBKEs (2). Section 2 provides some the basic facts and definitions concern Caputo fractional differential operators and Aboodh transom. We proved some theorems about Aboodh transom. In Section 3 we present the ARPST steps. Section 4 uses ARPST in getting the analytical solutions of Caputo WBKEs (2) with two cases, with perfect initial condition and with approximate initial condition. Section 5 presents some numerical solutions graphically and compares these solutions with other solutions of (2) that were obtained by Adomian decomposition Method (ADM) [34], the OHAM [35] and the Sumudu decomposition method (SDM) [36].

2 On Caputo operator

Consider I is an interval in $\mathbb{R}$ and τ is any continuous map on $I\times[0,\infty)$. Then the R-L derivative operator, [43], of $\tau(\nu_1,\nu_2)$ of order $\omega>0$ is defined by

$$\mathcal{D}^{\omega}_{\nu_2}\tau(\nu_1,\nu_2)=\partial^{n}_{\nu_2}\mathcal{I}^{\omega}_{\nu_2}\tau(\nu_1,\nu_2) \qquad n-1<\omega<n \tag{3}$$

where $n\in\mathbb{N}$ and $\mathcal{I}^{\omega}_{\nu_2}$ is the R-L integral operator, [44], of $\tau(\nu_1,\nu_2)$ of order ω is defined by

$$\mathcal{I}^{\omega}_{\nu_2}\tau(\nu_1,\nu_2)=\begin{cases}\dfrac{1}{\Gamma(\omega)}\displaystyle\int_0^{\nu_2}(\nu_2-\theta)^{\omega-1}\tau(\nu_1,\theta)d\theta, & \omega>0\\ \tau(\nu_1,\nu_2), & \omega=0.\end{cases} \tag{4}$$

The Caputo differential operator, [45], for $\tau(\nu_1,\nu_2)$ of order ω is defined by

$${}^{c}\mathcal{D}^{\omega}_{\nu_2}\tau(\nu_1,\nu_2)=\begin{cases}\mathcal{I}^{n-\omega}_{\nu_2}\partial^{n}_{\nu_2}\tau(\nu_1,\nu_2), & n-1<\omega<n\\ \partial^{\omega}_{\nu_2}\tau(\nu_1,\nu_2), & \omega\in\mathbb{N},\end{cases} \tag{5}$$

For $\nu_2\geq 0$ and for $n-1<\omega<n$, by [45] we have ${}^{c}\mathcal{D}^{\omega}_{\nu_2}\mathcal{I}^{\omega}_{\nu_2}\tau(\nu_1,\nu_2)=\tau(\nu_1,\nu_2)$, ${}^{c}\mathcal{D}^{\omega}_{\nu_2}\nu_2^{\gamma}=\frac{\Gamma(\omega+2)}{\Gamma(\gamma-\omega+2)}\nu_2^{\gamma-\omega}$ and

$$\mathcal{I}^{\omega}_{\nu_2}\,{}^{c}\mathcal{D}^{\omega}_{\nu_2}\tau(\nu_1,\nu_2)=\tau(\nu_1,\nu_2)+\sum_{k=0}^{n-1}\frac{\nu_2^k}{k!}\partial^{k}_{\nu_2}\tau(\nu_1,\nu_2)\Big|_{\nu_2=0},$$

where $\gamma > -1$, $c \in \mathbb{R}$ is a constant and $n \in \mathbb{N}$.

Let $\tau(\nu_1, \nu_2)$ be a map and of exponential order ρ on $I \times [0, \infty)$. The Aboodh transform $A_{\nu_2\sigma}$, [29], of $\tau(\nu_1, \nu_2)$ according to ν_2 is recalled by

$$\tau^{A\nu_2}(\nu_1, \sigma) := A_{\nu_2\sigma}\tau(\nu_1, \nu_2) = \frac{1}{\sigma}\int_0^\infty e^{-\nu_2\sigma}\tau(\nu_1, \nu_2)d\nu_2 \quad \rho_1 \leq \sigma \leq \rho_2. \tag{6}$$

The inverse Aboodh transform $A^{-1}_{\nu_2\sigma}$ of $\tau^{A\nu_2}(\nu_1, \sigma)$ according to σ is given by

$$\tau(\nu_1, \nu_2) := A^{-1}_{\nu_2\sigma}\tau^{A\nu_2}(\nu_1, \sigma) = \frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} \sigma e^{\nu_2\sigma}\tau^{A\nu_2}(\nu_1, \sigma)d\sigma. \tag{7}$$

Lemma 2.1. [32] Let $\tau, \pi : I \times [0, \infty) \to \mathbb{R}$ be two maps on $I \times [0, \infty)$. Then

1. $A_{\nu_2\sigma}[c_1\tau(\nu_1, \nu_2) + c_2\pi(\nu_1, \nu_2)] = c_1A_{\nu_2\sigma}\tau(\nu_1, \nu_2) + c_2A_{\nu_2\sigma}\pi(\nu_1, \nu_2)$, where c_1 and c_2 are constants;

2. $A^{-1}_{\nu_2\sigma}[c_1A_{\nu_2\sigma}\tau(\nu_1, \nu_2) + c_2A_{\nu_2\sigma}\pi(\nu_1, \nu_2)] = c_1\tau(\nu_1, \nu_2) + c_2\pi(\nu_1, \nu_2)$, where c_1 and c_2 are constants;

3. $A_{\nu_2\sigma}\mathcal{I}^{\omega}_{\nu_2}\tau(\nu_1, \nu_2) = \sigma^{-\omega}\tau^{A\nu_2}(\nu_1, \sigma)$;

4. $A_{\nu_2\sigma}\left[{}^c\mathcal{D}^{\omega}_{\nu_2}\tau(\nu_1, \nu_2)\right] = \sigma^{\omega}\tau^{A\nu_2}(\nu_1, \sigma) - \sum_{j=0}^{n-1}\sigma^{\omega-j-2}\partial^j_{\nu_2}\tau(\nu_1, \nu_2)\Big|_{\nu_2=0} \quad (n - 1 < \omega < n)$.

Recall [33] that the form of the power series is presented by

$$\sum_{j=0}^{\infty}\tau_j(\nu_1)(\nu_2 - \nu_{20})^{jn} = \tau_0(\nu_1)(\nu_2 - \nu_{20})^0 + \tau_1(\nu_1)(\nu_2 - \nu_{20})^n + \tau_2(\nu_1)(\nu_2 - \nu_{20})^{2n} + \cdots \tag{8}$$

where $\nu_1 = (\nu_{11}, \nu_{12}, \cdots, \nu_{1n}) \in \mathbb{R}^n, n \in \mathbb{N}$. The MFPS denotes the multiple fractional of power series about ν_{20} with ν_2 and the terms $\tau_j(\nu_1)$.

Lemma 2.2. Let τ be a map of $I \times [0, \infty)$ into $\mathbb{R}$. Then

$$A_{\nu_2\sigma}\left[{}^c\mathcal{D}^{k\omega}_{\nu_2}\tau(\nu_1, \nu_2)\right] = \sigma^{k\omega}\tau^{A\nu_2}(\nu_1, \sigma) - \sum_{j=0}^{k-1}\sigma^{(k-j)\omega-2}\,{}^c\mathcal{D}^{j\omega}_{\nu_2}\tau(\nu_1, \nu_2)\Big|_{\nu_2=0} (0 < \omega \leq 1). \tag{9}$$

Proof. To prove the relation (9), we will use the induction. If $k = 1$ and since $0 < \omega \leq 1$ in (9) then from Lemma above we have

$$A_{\nu_2\sigma}\left[{}^c\mathcal{D}^{\omega}_{\nu_2}\tau(\nu_1, \nu_2)\right] = \sigma^{\omega}\tau^{A\nu_2}(\nu_1, \sigma) - \sigma^{\omega-2}\tau(\nu_1, 0). \tag{10}$$

If $k = 2$, let $\pi(\nu_1, \nu_2) = {}^c\mathcal{D}^{\omega}_{\nu_2}\tau(\nu_1, \nu_2)$. By (10) above we have

$$\begin{aligned}
&A_{\nu_2\sigma}\left[{}^c\mathcal{D}^{\omega}_{\nu_2}\pi(\nu_1, \nu_2)\right] = \sigma^{\omega}\pi^{A\nu_2}(\nu_1, \sigma) - \sigma^{\omega-2}\pi(\nu_1, 0)\\
&= \sigma^{\omega}A_{\nu_2\sigma}{}^c\mathcal{D}^{\omega}_{\nu_2}\tau(\nu_1, \nu_2) - \sigma^{\omega-2}\,{}^c\mathcal{D}^{\omega}_{\nu_2}\tau(\nu_1, \nu_2)\Big|_{\nu_2=0}\\
&= \sigma^{\omega}\left[\sigma^{\omega}\tau^{A\nu_2}(\nu_1, \sigma) - \sigma^{\omega-2}\tau(\nu_1, 0)\right] - \sigma^{\omega-2}\,{}^c\mathcal{D}^{\omega}_{\nu_2}\tau(\nu_1, \nu_2)\Big|_{\nu_2=0}\\
&= \sigma^{2\omega}\tau^{A\nu_2}(\nu_1, \sigma) - \sigma^{2\omega-2}\tau(\nu_1, 0) - \sigma^{\omega-2}\,{}^c\mathcal{D}^{\omega}_{\nu_2}\tau(\nu_1, \nu_2)\Big|_{\nu_2=0}.
\end{aligned} \tag{11}$$

If (9) is true at $k = m$ then we have

$$A_{\nu_2\sigma}\left[{}^{c}\mathcal{D}_{\nu_2}^{m\omega}\tau(\nu_1,\nu_2)\right] = \sigma^{m\omega}\tau^{A\nu_2}(\nu_1,\sigma) - \sum_{j=0}^{m-1}\sigma^{(m-j)\omega-2}\, {}^{c}\mathcal{D}_{\nu_2}^{j\omega}\tau(\nu_1,\nu_2)\Big|_{\nu_2=0}. \tag{12}$$

We show that (9) holds at $k = m+1$. Let $\pi(\nu_1,\nu_2) = {}^{c}\mathcal{D}_{\nu_2}^{m\omega}\tau(\nu_1,\nu_2)$. By (10) and (12) we have

$$\begin{aligned}
&A_{\nu_2\sigma}\left[{}^{c}\mathcal{D}_{\nu_2}^{k\omega}\pi(\nu_1,\nu_2)\right] = A_{\nu_2\sigma}\left[{}^{c}\mathcal{D}_{\nu_2}^{\omega}\pi(\nu_1,\nu_2)\right] = \sigma^{\omega}\pi^{A\nu_2}(\nu_1,\sigma) - \sigma^{\omega-2}\pi(\nu_1,0)\\
&= \sigma^{\omega}A_{\nu_2\sigma}{}^{c}\mathcal{D}_{\nu_2}^{m\omega}\tau(\nu_1,\nu_2) - \sigma^{\omega-2}\, {}^{c}\mathcal{D}_{\nu_2}^{m\omega}\tau(\nu_1,\nu_2)\Big|_{\nu_2=0}\\
&= \sigma^{\omega}\left[\sigma^{m\omega}\tau^{A\nu_2}(\nu_1,\sigma) - \sum_{j=0}^{m-1}\sigma^{(m-j)\omega-2}\, {}^{c}\mathcal{D}_{\nu_2}^{j\omega}\tau(\nu_1,\nu_2)\Big|_{\nu_2=0}\right]\\
&- \sigma^{\omega-2}\, {}^{c}\mathcal{D}_{\nu_2}^{m\omega}\tau(\nu_1,\nu_2)\Big|_{\nu_2=0}\\
&= \sigma^{(m+1)\omega}\tau^{A\nu_2}(\nu_1,\sigma) - \sum_{j=0}^{m-1}\sigma^{(m+1-j)\omega-2}\, {}^{c}\mathcal{D}_{\nu_2}^{j\omega}\tau(\nu_1,\nu_2)\Big|_{\nu_2=0}\\
&- \sigma^{\omega-2}\, {}^{c}\mathcal{D}_{\nu_2}^{\omega}\tau(\nu_1,\nu_2)\Big|_{\nu_2=0}\\
&= \sigma^{(m+1)\omega}\tau^{A\nu_2}(\nu_1,\sigma) - \sum_{j=0}^{m}\sigma^{(m+1-j)\omega-2}\, {}^{c}\mathcal{D}_{\nu_2}^{j\omega}\tau(\nu_1,\nu_2)\Big|_{\nu_2=0}\\
&= \sigma^{k\omega}\tau^{A\nu_2}(\nu_1,\sigma) - \sum_{j=0}^{k-1}\sigma^{(k-j)\omega-2}\, {}^{c}\mathcal{D}_{\nu_2}^{j\omega}\tau(\nu_1,\nu_2)\Big|_{\nu_2=0}
\end{aligned} \tag{13}$$

Then (9) is true. □

Lemma 2.3. Let τ be an order-exponential map of $\mathbb{R}^n \times [0,\infty)$ into $\mathbb{R}$. The MFPS notation for $\tau^{A\nu_2}$ is given by

$$\tau^{A\nu_2}(\nu_1,\sigma) = \sum_{j=0}^{\infty}\frac{f_j(\nu_1)}{\sigma^{jn+2}} \qquad \sigma > 0, \tag{14}$$

where $\nu_1 = (\nu_{11},\nu_{12},\cdots,\nu_{1n}) \in \mathbb{R}^n, n \in \mathbb{N}$.

Proof. By using the Taylor series we get

$$\tau(\nu_1,\nu_2) = f_0(\nu_1) + f_1(\nu_1)\frac{\nu_2^n}{\Gamma(n+1)} + f_2(\nu_1)\frac{\nu_2^{2n}}{\Gamma(2n+1)} + \cdots \tag{15}$$

Hence

$$\begin{aligned}
&A_{\nu_2\sigma}\tau(\nu_1,\nu_2) = A_{\nu_2\sigma}f_0(\nu_1) + f_1(\nu_1)\frac{A_{\nu_2\sigma}\nu_2^n}{\Gamma(n+1)} + f_2(\nu_1)\frac{A_{\nu_2\sigma}\nu_2^{2n}}{\Gamma(2n+1)} + \cdots\\
&= \frac{f_0(\nu_1)}{\sigma^2} + f_1(\nu_1)\frac{\Gamma(n+1)}{\sigma^{n+2}\Gamma(n+1)} + f_2(\nu_1)\frac{\Gamma(2n+1)}{\sigma^{2n+2}\Gamma(2n+1)} + \cdots\\
&= \sum_{j=0}^{\infty}\frac{f_j(\nu_1)}{\sigma^{jn+2}}
\end{aligned} \tag{16}$$

□

Lemma 2.4. Let τ be an order-exponential map of $\mathbb{R}^n \times [0,\infty)$ into $\mathbb{R}$. Then

$$\lim_{\sigma\to\infty} \sigma^2 \tau^{A\nu_2}(\nu_1,\sigma) = \tau(\nu_1,0) \text{ for all} \nu_1 \in \mathbb{R}^n.$$

Proof. Put $\tau(\nu_1,0) = f_0(\nu_1)$ to obtain the desired. □

Theorem 2.5. Let τ be an order-exponential map of $\mathbb{R}^n \times [0,\infty)$ into $\mathbb{R}$. Then

$$\tau^{A\nu_2}(\nu_1,\sigma) = \sum_{j=0}^{\infty} \frac{f_j(\nu_1)}{\sigma^{jn+2}} \qquad (0 < \omega \le 1) \tag{17}$$

for all $\nu_1 \in \mathbb{R}^n$ and $\sigma > 0$, where $f_j(\nu_1) = \partial_{\nu_2}^{jn} \tau(\nu_1,\nu_2)\Big|_{\nu_2=0}$.

Proof. The form of Taylor's series is given by

$$f_1(\nu_1) = \sigma^{n+2}\tau^{A\nu_2}(\nu_1,\sigma) - \sigma^n f_0(\nu_1) - \frac{1}{\sigma^n} f_2(\nu_1) - \frac{1}{\sigma^{2n}} f_3(\nu_1) - \cdots \tag{18}$$

Use the limit of (18) at $\sigma \to \infty$

$$f_1(\nu_1) = \lim_{\sigma\to\infty} \left[\sigma^{n+2}\tau^{A\nu_2}(\nu_1,\sigma) - \sigma^n f_0(\nu_1)\right]. \tag{19}$$

By Lemma 2.2

$$f_1(\nu_1) = \lim_{\sigma\to\infty} \sigma^2 A_{\nu_2\sigma}\left[\partial_{\nu_2}^n \tau(\nu_1,\nu_2)\right]. \tag{20}$$

By Lemma 2.4 we have $f_1(\nu_1) = \partial_{\nu_2}^n \tau(\nu_1,0)$. Similar, the form of Taylor's series of f_2 will be

$$f_2(\nu_1) = \sigma^{2n+2}\tau^{A\nu_2}(\nu_1,\sigma) - \sigma^{2n} f_0(\nu_1) - \sigma^n f_1(\nu_1) - \frac{1}{\sigma^n} f_3(\nu_1) - \frac{1}{\sigma^{2n}} f_4(\nu_1) - \cdots \tag{21}$$

Take the limit of (21) when $\sigma \to \infty$ to obtain

$$f_2(\nu_1) = \lim_{\sigma\to\infty} \left[\sigma^{2n+2}\tau^{A\nu_2}(\nu_1,\sigma) - \sigma^{2n} f_0(\nu_1) - \sigma^n f_1(\nu_1)\right]. \tag{22}$$

By Lemmas 2.2 and 2.4 we have $f_2(\nu_1) = \partial_{\nu_2}^{2n} \tau(\nu_1,0)$. By the continuity, we indicate that $f_j(\nu_1) = \partial_{\nu_2}^{jn} \tau(\nu_1,0)$. □

In theorem above we get

$$A_{\nu_2\sigma}\left[{}^c\mathcal{D}_{\nu_2}^{\omega} \tau(\nu_1,\nu_2)\right] = \sum_{j=0}^{\infty} \frac{1}{\sigma^{j\omega+2}} {}^c\mathcal{D}_{\nu_2}^{j\omega} \tau(\nu_1,\nu_2)\Big|_{\nu_2=0} \qquad (0 < \omega \le 1)$$

for all $\nu_1 \in \mathbb{R}^n$, $\sigma > 0$ and the inverse Aboodh transform will be

$$\tau(\nu_1,\nu_2) = \sum_{j=0}^{\infty} \frac{{}^c\mathcal{D}_{\nu_2}^{j\omega} \tau(\nu_1,\nu_2)\Big|_{\nu_2=0}}{\Gamma(j\omega+2)} \; \nu_2^{j\omega} \qquad (0 < \omega \le 1)$$

for all $\nu_1 \in \mathbb{R}^n$ and $\nu_2 > 0$.

Theorem 2.6. Let τ be an order-exponential map of $\mathbb{R}^n \times [0, \infty)$ into $\mathbb{R}$. If

$$\left|\sigma^r A_{\nu_2\sigma}\left[{}^c\mathcal{D}_{\nu_2}^{(n+1)\omega}\tau(\nu_1,\nu_2)\right]\right| \leq M$$

for all $0 < \sigma \leq q$ and $0 < \omega \leq 1$ then the residual $\mathcal{R}es_n(\nu_1, \sigma)$ of MFPS satisfies

$$\|\mathcal{R}es_n(\nu_1,\sigma)\| \leq \frac{M}{\sigma^{(n+1)\omega}}.$$

Proof. Use form of Taylor's series as

$$\mathcal{R}es_n(\nu_1,\sigma) = \tau^{A\nu_2}(\nu_1,\sigma) + \sum_{j=0}^{n} \frac{f_j(\nu_1)}{\sigma^{j\omega+2}}. \tag{23}$$

By Theorem 2.5 we have

$$\mathcal{R}es_n(\nu_1,\sigma) = \tau^{A\nu_2}(\nu_1,\sigma) + \sum_{j=0}^{n} \frac{{}^c\mathcal{D}_{\nu_2}^{j\omega}\tau(\nu_1,0)}{\sigma^{j\omega+2}}. \tag{24}$$

Multiply (24) by $\sigma^{(n+1)\omega}$ to get

$$\sigma^{(n+1)\omega}\mathcal{R}es_n(\nu_1,\sigma) = \sigma^{(n+1)\omega}\tau^{A\nu_2}(\nu_1,\sigma) + \sum_{j=0}^{n} \sigma^{(n+1-j)\omega-2c}\mathcal{D}_{\nu_2}^{j\omega}\tau(\nu_1,0). \tag{25}$$

By Lemma 2.2 we have

$$\sigma^{(n+1)\omega}\mathcal{R}es_n(\nu_1,\sigma) = A_{\nu_2\sigma}\left[{}^c\mathcal{D}_{\nu_2}^{(n+1)\omega}\tau(\nu_1,\nu_2)\right]. \tag{26}$$

Hence

$$\left|\sigma^{(n+1)\omega}\mathcal{R}es_n(\nu_1,\sigma)\right| = \left|A_{\nu_2\sigma}\left[{}^c\mathcal{D}_{\nu_2}^{(n+1)\omega}\tau(\nu_1,\nu_2)\right]\right|.$$

That is, $|\mathcal{R}es_n(\nu_1,\sigma)| \leq \dfrac{M}{\sigma^{(n+1)\omega}}$. □

3 The ARPST steps

Consider Caputo WBKEs :

$$\begin{cases} {}^c\mathcal{D}_{\nu_2}^{\omega}\tau(\nu_1,\nu_2) + \tau(\nu_1,\nu_2)\partial_{\nu_1}\tau(\nu_1,\nu_2) + \partial_{\nu_1}\pi(\nu_1,\nu_2) + \beta\partial_{\nu_1}^2\tau(\nu_1,\nu_2) = 0 \\ {}^c\mathcal{D}_{\nu_2}^{\omega}\pi(\nu_1,\nu_2) + \partial_{\nu_1}\left[\tau(\nu_1,\nu_2)\pi(\nu_1,\nu_2)\right] - \beta\partial_{\nu_1}^2\pi(\nu_1,\nu_2) + \beta'\partial_{\nu_1}^3\tau(\nu_1,\nu_2) = 0 \\ \tau(\nu_1,0) = u_0(\nu_1), \quad \pi(\nu_1,0) = v_0(\nu_1) \end{cases} \tag{27}$$

where $0 < \omega \leq 1$. Take the Aboodh transform $A_{\nu_2\sigma}$ of (27)

$$\begin{aligned} &A_{\nu_2\sigma}\left[{}^c\mathcal{D}_{\nu_2}^{\omega}\tau(\nu_1,\nu_2)\right] = -A_{\nu_2\sigma}\left[\tau(\nu_1,\nu_2)\partial_{\nu_1}\tau(\nu_1,\nu_2)\right] - A_{\nu_2\sigma}\left[\partial_{\nu_1}\pi(\nu_1,\nu_2)\right] \\ &- \beta A_{\nu_2\sigma}\left[\partial_{\nu_1}^2\tau(\nu_1,\nu_2)\right] \\ &A_{\nu_2\sigma}\left[{}^c\mathcal{D}_{\nu_2}^{\omega}\pi(\nu_1,\nu_2)\right] = -A_{\nu_2\sigma}\left[\partial_{\nu_1}\left[\tau(\nu_1,\nu_2)\pi(\nu_1,\nu_2)\right]\right] + \beta A_{\nu_2\sigma}\left[\partial_{\nu_1}^2\pi(\nu_1,\nu_2)\right] \\ &- \beta' A_{\nu_2\sigma}\left[\partial_{\nu_1}^3\tau(\nu_1,\nu_2)\right] \end{aligned} \tag{28}$$

By Lemma 2.1, $A_{\nu_2\sigma}$ for the left side of (28) will be

$$\begin{aligned} A_{\nu_2\sigma}\left[{}^{c}\mathcal{D}^{\omega}_{\nu_2}\tau(\nu_1,\nu_2)\right] &= \sigma^{\omega}\tau^{A\nu_2}(\nu_1,\sigma)-\sigma^{\omega-2}u_0(\nu_1)\\ A_{\nu_2\sigma}\left[{}^{c}\mathcal{D}^{\omega}_{\nu_2}\pi(\nu_1,\nu_2)\right] &= \sigma^{\omega}\pi^{A\nu_2}(\nu_1,\sigma)-\sigma^{\omega-2}v_0(\nu_1). \end{aligned} \tag{29}$$

Then from (28) and (29) we get

$$\begin{aligned} \tau^{A\nu_2}(\nu_1,\sigma) &= \frac{1}{\sigma^2}u_0(\nu_1)-\frac{1}{\sigma^{\omega}}A_{\nu_2\sigma}\left[A^{-1}_{\nu_2\sigma}(\tau^{A\nu_2}(\nu_1,\sigma))\partial_{\nu_1}A^{-1}_{\nu_2\sigma}(\tau^{A\nu_2}(\nu_1,\sigma))\right]\\ &-\frac{1}{\sigma^{\omega}}A_{\nu_2\sigma}\left[\partial_{\nu_1}A^{-1}_{\nu_2\sigma}(\pi^{A\nu_2}(\nu_1,\sigma))\right]-\frac{\beta}{\sigma^{\omega}}A_{\nu_2\sigma}\left[\partial^2_{\nu_1}A^{-1}_{\nu_2\sigma}(\tau^{A\nu_2}(\nu_1,\sigma))\right]\\ \pi^{A\nu_2}(\nu_1,\sigma) &= \frac{1}{\sigma^2}v_0(\nu_1)-\frac{1}{\sigma^{\omega}}A_{\nu_2\sigma}\left[\partial_{\nu_1}\left[A^{-1}_{\nu_2\sigma}(\pi^{A\nu_2}(\nu_1,\sigma))A^{-1}_{\nu_2\sigma}(\tau^{A\nu_2}(\nu_1,\sigma))\right]\right]\\ &+\frac{\beta}{\sigma^{\omega}}A_{\nu_2\sigma}\left[\partial^2_{\nu_1}A^{-1}_{\nu_2\sigma}(\pi^{A\nu_2}(\nu_1,\sigma))\right]-\frac{\beta'}{\sigma^{\omega}}A_{\nu_2\sigma}\left[\partial^3_{\nu_1}A^{-1}_{\nu_2\sigma}(\tau^{A\nu_2}(\nu_1,\sigma))\right]. \end{aligned} \tag{30}$$

The analytical solutions $\tau^{A\nu_2}(\nu_1,\sigma)$ and $\pi^{A\nu_2}(\nu_1,\sigma)$ for (28) are given by

$$\begin{aligned} \tau^{A\nu_2}(\nu_1,\sigma) &= \frac{1}{\sigma^2}u_0(\nu_1)+\sum_{j=1}^{\infty}\frac{u_j(\nu_1)}{\sigma^{j\omega+2}}\\ \pi^{A\nu_2}(\nu_1,\sigma) &= \frac{1}{\sigma^2}v_0(\nu_1)+\sum_{j=1}^{\infty}\frac{v_j(\nu_1)}{\sigma^{j\omega+2}}. \end{aligned} \tag{31}$$

By Lemma 2.1 and the initial condition in (27), we get that the partial sums sequences $\left\langle\tau^{A\nu_2}_n\right\rangle_{n\in\mathbb{N}\cup\{0\}}$ and $\left\langle\pi^{A\nu_2}_n\right\rangle_{n\in\mathbb{N}\cup\{0\}}$ of (31) as solutions of

$$\begin{aligned} \tau^{A\nu_2}_n(\nu_1,\sigma) &= \frac{1}{\sigma^2}u_0(\nu_1)+\sum_{j=1}^{n}\frac{u_j(\nu_1)}{\sigma^{j\omega+2}}\\ \pi^{A\nu_2}_n(\nu_1,\sigma) &= \frac{1}{\sigma^2}v_0(\nu_1)+\sum_{j=1}^{n}\frac{v_j(\nu_1)}{\sigma^{j\omega+2}}. \end{aligned} \tag{32}$$

Construct the residual maps of Aboodh transform, $A_{\nu_2\sigma}\mathcal{R}es\ \tau_{\nu_1\nu_2}$ and $A_{\nu_2\sigma}\mathcal{R}es\ \pi_{\nu_1\nu_2}$ for (30) as follow:

$$\begin{aligned} A_{\nu_2\sigma}\mathcal{R}es\ \tau_{\nu_1\nu_2} &= \tau^{A\nu_2}(\nu_1,\sigma)-\frac{1}{\sigma^2}u_0(\nu_1)+\frac{1}{\sigma^{\omega}}A_{\nu_2\sigma}\left[A^{-1}_{\nu_2\sigma}(\tau^{A\nu_2}(\nu_1,\sigma))\partial_{\nu_1}A^{-1}_{\nu_2\sigma}(\tau^{A\nu_2}(\nu_1,\sigma))\right]\\ &+\frac{1}{\sigma^{\omega}}A_{\nu_2\sigma}\left[\partial_{\nu_1}A^{-1}_{\nu_2\sigma}(\pi^{A\nu_2}(\nu_1,\sigma))\right]+\frac{\beta}{\sigma^{\omega}}A_{\nu_2\sigma}\left[\partial^2_{\nu_1}A^{-1}_{\nu_2\sigma}(\tau^{A\nu_2}(\nu_1,\sigma))\right]\\ A_{\nu_2\sigma}\mathcal{R}es\ \pi_{\nu_1\nu_2} &= \pi^{A\nu_2}(\nu_1,\sigma)-\frac{1}{\sigma^2}v_0(\nu_1)+\frac{1}{\sigma^{\omega}}A_{\nu_2\sigma}\left[\partial_{\nu_1}\left[A^{-1}_{\nu_2\sigma}(\pi^{A\nu_2}(\nu_1,\sigma))A^{-1}_{\nu_2\sigma}(\tau^{A\nu_2}(\nu_1,\sigma))\right]\right]\\ &-\frac{\beta}{\sigma^{\omega}}A_{\nu_2\sigma}\left[\partial^2_{\nu_1}A^{-1}_{\nu_2\sigma}(\pi^{A\nu_2}(\nu_1,\sigma))\right]+\frac{\beta'}{\sigma^{\omega}}A_{\nu_2\sigma}\left[\partial^3_{\nu_1}A^{-1}_{\nu_2\sigma}(\tau^{A\nu_2}(\nu_1,\sigma))\right] \end{aligned}$$

and the n-th Aboodh residual functions $A_{\nu_2\sigma}\mathcal{R}es_n\ \tau_{\nu_1\nu_2}$ are

$$\begin{cases} A_{\nu_2\sigma}\mathcal{R}es_n\ \tau_{\nu_1\nu_2} = \tau^{A\nu_2}_n(\nu_1,\sigma)-\frac{1}{\sigma^2}u_0(\nu_1)+\frac{1}{\sigma^{\omega}}A_{\nu_2\sigma}\left[A^{-1}_{\nu_2\sigma}(\tau^{A\nu_2}_n(\nu_1,\sigma))\partial_{\nu_1}A^{-1}_{\nu_2\sigma}(\tau^{A\nu_2}_n(\nu_1,\sigma))\right]\\ +\frac{1}{\sigma^{\omega}}A_{\nu_2\sigma}\left[\partial_{\nu_1}A^{-1}_{\nu_2\sigma}(\pi^{A\nu_2}_n(\nu_1,\sigma))\right]+\frac{\beta}{\sigma^{\omega}}A_{\nu_2\sigma}\left[\partial^2_{\nu_1}A^{-1}_{\nu_2\sigma}(\tau^{A\nu_2}_n(\nu_1,\sigma))\right]\\ A_{\nu_2\sigma}\mathcal{R}es_n\ \pi_{\nu_1\nu_2} = \pi^{A\nu_2}_n(\nu_1,\sigma)-\frac{1}{\sigma^2}v_0(\nu_1)+\frac{1}{\sigma^{\omega}}A_{\nu_2\sigma}\left[\partial_{\nu_1}\left[A^{-1}_{\nu_2\sigma}(\pi^{A\nu_2}_n(\nu_1,\sigma))A^{-1}_{\nu_2\sigma}(\tau^{A\nu_2}_n(\nu_1,\sigma))\right]\right]\\ -\frac{\beta}{\sigma^{\omega}}A_{\nu_2\sigma}\left[\partial^2_{\nu_1}A^{-1}_{\nu_2\sigma}(\pi^{A\nu_2}_n(\nu_1,\sigma))\right]+\frac{\beta'}{\sigma^{\omega}}A_{\nu_2\sigma}\left[\partial^3_{\nu_1}A^{-1}_{\nu_2\sigma}(\tau^{A\nu_2}_n(\nu_1,\sigma))\right] \end{cases}$$

It is clear that $A_{\nu_2\sigma}\,\mathrm{Res}\ \tau_{\nu_1\nu_2}=0$ and $A_{\nu_2\sigma}\,\mathrm{Res}\ \pi_{\nu_1\nu_2}=0$. Also, we have

$$\lim_{n\to\infty} A_{\nu_2\sigma}\,\mathrm{Res}_n\ \tau_{\nu_1\nu_2}=A_{\nu_2\sigma}\,\mathrm{Res}\ \tau_{\nu_1\nu_2} \quad \text{and} \quad \lim_{n\to\infty} A_{\nu_2\sigma}\,\mathrm{Res}_n\ \pi_{\nu_1\nu_2}=A_{\nu_2\sigma}\,\mathrm{Res}\ \pi_{\nu_1\nu_2}$$

for all $\sigma>0$. If

$$\lim_{\sigma\to\infty}\sigma^2 A_{\nu_2\sigma}\,\mathrm{Res}\ \tau_{\nu_1\nu_2}=0 \quad \text{and} \quad \lim_{\sigma\to\infty}\sigma^2 A_{\nu_2\sigma}\,\mathrm{Res}\ \pi_{\nu_1\nu_2}=0$$

then

$$\lim_{\sigma\to\infty}\sigma^2 A_{\nu_2\sigma}\,\mathrm{Res}_n\ \tau_{\nu_1\nu_2}=0 \quad \text{and} \quad \lim_{\sigma\to\infty}\sigma^2 A_{\nu_2\sigma}\,\mathrm{Res}_n\ \pi_{\nu_1\nu_2}=0.$$

In general, if

$$\lim_{\sigma\to\infty}\sigma^{n\omega+2} A_{\nu_2\sigma}\,\mathrm{Res}\ \tau_{\nu_1\nu_2}=0 \quad \text{and} \quad \lim_{\sigma\to\infty}\sigma^{n\omega+2} A_{\nu_2\sigma}\,\mathrm{Res}\ \pi_{\nu_1\nu_2}=0,$$

then

$$\lim_{\sigma\to\infty}\sigma^{n\omega+2} A_{\nu_2\sigma}\,\mathrm{Res}_n\ \tau_{\nu_1\nu_2}=0 \quad \text{and} \quad \lim_{\sigma\to\infty}\sigma^{n\omega+2} A_{\nu_2\sigma}\,\mathrm{Res}_n\ \pi_{\nu_1\nu_2}=0$$

for all $n=1,2,3,\cdots$ and $0<\omega\le 1$. To calculate the coefficients $u_n(\nu_1)$ and $v_n(\nu_1)$, we use the iterative method in the following:

$$\lim_{\sigma\to\infty}\sigma^{n\omega+2} A_{\nu_2\sigma}\mathcal{R}es_n\ \tau_{\nu_1\nu_2}=0 \text{ and } \lim_{\sigma\to\infty}\sigma^{n\omega+2} A_{\nu_2\sigma}\mathcal{R}es_n\ \pi_{\nu_1\nu_2}=0 \tag{33}$$

for all $n=1,2,3,\cdots$. Put the coefficients, $u_n(\nu_1)$ and $v_n(\nu_1)$ in (32) to get the n-th proximate solutions $\tau_n^{A\nu_2}(\nu_1,\sigma)$ and $\pi_n^{A\nu_2}(\nu_1,\sigma)$ of (30). Finally, by using the inverse Aboodh transform of $\tau_n^{A\nu_2}(\nu_1,\sigma)$ and $\pi_n^{A\nu_2}(\nu_1,\sigma)$ we get the n-th proximate solutions $\tau_{\nu_2}^n(\nu_1,\nu_2)$ and $\pi_{\nu_2}^n(\nu_1,\nu_2)$ of (27).

4 Some applications

In this section we use ARPST above to get analytical solutions of Caputo WBKEs (2) where $\beta=0$ and $\beta'=1$. Firstly, we use ARPST to get the solution with the approximate initial condition that obtained via the Taylor series for third order. Consider the following Caputo WBKEs

$$\begin{cases} {}^c\mathcal{D}^{\omega}_{\nu_2}\tau(\nu_1,\nu_2)+\tau(\nu_1,\nu_2)\partial_{\nu_1}\tau(\nu_1,\nu_2)+\partial_{\nu_1}\pi(\nu_1,\nu_2)=0\\ {}^c\mathcal{D}^{\omega}_{\nu_2}\pi(\nu_1,\nu_2)+\partial_{\nu_1}\left[\tau(\nu_1,\nu_2)\pi(\nu_1,\nu_2)\right]+\partial^3_{\nu_1}\tau(\nu_1,\nu_2)=0\\ \tau(\nu_1,0)=-0.2576+0.01448\nu_1-0.0019\nu_1^2\\ \pi(\nu_1,0)=-0.01448+0.0038\nu_1-0.00144\nu_1^2. \end{cases} \tag{34}$$

Take Aboodh transforms on (34) to obtain

$$\begin{aligned} &\tau^{A\nu_2}(\nu_1,\sigma)=\frac{1}{\sigma^2}\left[-0.2576+0.01448\nu_1-0.0019\nu_1^2\right]\\ &-\frac{1}{\sigma^\omega}A_{\nu_2\sigma}\left[A^{-1}_{\nu_2\sigma}(\tau^{A\nu_2}(\nu_1,\sigma))\partial_{\nu_1}A^{-1}_{\nu_2\sigma}(\tau^{A\nu_2}(\nu_1,\sigma))\right]-\frac{1}{\sigma^\omega}A_{\nu_2\sigma}\left[\partial_{\nu_1}A^{-1}_{\nu_2\sigma}(\pi^{A\nu_2}(\nu_1,\sigma))\right]\\ &\pi^{A\nu_2}(\nu_1,\sigma)=\frac{1}{\sigma^2}\left[-0.01448+0.0038\nu_1-0.00144\nu_1^2\right]\\ &-\frac{1}{\sigma^\omega}A_{\nu_2\sigma}\left[\partial_{\nu_1}\left[A^{-1}_{\nu_2\sigma}(\pi^{A\nu_2}(\nu_1,\sigma))A^{-1}_{\nu_2\sigma}(\tau^{A\nu_2}(\nu_1,\sigma))\right]\right]-\frac{1}{\sigma^\omega}A_{\nu_2\sigma}\left[\partial^3_{\nu_1}A^{-1}_{\nu_2\sigma}(\tau^{A\nu_2}(\nu_1,\sigma))\right]. \end{aligned} \tag{35}$$

and the partial sums sequences $\left\langle \tau_n^{A\nu_2}\right\rangle_{n\in\mathbb{N}\cup\{0\}}$ and $\left\langle \pi_n^{A\nu_2}\right\rangle_{n\in\mathbb{N}\cup\{0\}}$ are given by

$$\begin{aligned} \tau_n^{A\nu_2}(\nu_1,\sigma) &= \frac{1}{\sigma^2}\left[-0.2576+0.01448\nu_1-0.0019\nu_1^2\right]+\sum_{j=1}^{n}\frac{u_j(\nu_1)}{\sigma^{j\omega+2}} \\ \pi_n^{A\nu_2}(\nu_1,\sigma) &= \frac{1}{\sigma^2}\left[-0.01448+0.0038\nu_1-0.00144\nu_1^2\right]+\sum_{j=1}^{n}\frac{v_j(\nu_1)}{\sigma^{j\omega+2}}. \end{aligned} \tag{36}$$

The n-th constructions of (35) are given by

$$\begin{aligned} &\tau_n^{A\nu_2}(\nu_1,\sigma) = \frac{1}{\sigma^2}\left[-0.2576+0.01448\nu_1-0.0019\nu_1^2\right] \\ &-\frac{1}{\sigma^\omega}A_{\nu_2\sigma}\left[A_{\nu_2\sigma}^{-1}(\tau_n^{A\nu_2}(\nu_1,\sigma))\partial_{\nu_1}A_{\nu_2\sigma}^{-1}(\tau_n^{A\nu_2}(\nu_1,\sigma))\right]-\frac{1}{\sigma^\omega}A_{\nu_2\sigma}\left[\partial_{\nu_1}A_{\nu_2\sigma}^{-1}(\pi_n^{A\nu_2}(\nu_1,\sigma))\right] \\ &\pi_n^{A\nu_2}(\nu_1,\sigma) = \frac{1}{\sigma^2}\left[-0.01448+0.0038\nu_1-0.00144\nu_1^2\right] \\ &-\frac{1}{\sigma^\omega}A_{\nu_2\sigma}\left[\partial_{\nu_1}\left[A_{\nu_2\sigma}^{-1}(\pi_n^{A\nu_2}(\nu_1,\sigma))A_{\nu_2\sigma}^{-1}(\tau_n^{A\nu_2}(\nu_1,\sigma))\right]\right] \\ &-\frac{1}{\sigma^\omega}A_{\nu_2\sigma}\left[\partial_{\nu_1}^3A_{\nu_2\sigma}^{-1}(\tau_n^{A\nu_2}(\nu_1,\sigma))\right]. \end{aligned} \tag{37}$$

for $n=1,2,3,\cdots$. The Aboodh residual maps $A_{\nu_2\sigma}\mathcal{R}es\ \tau(\nu_1,\sigma)$ and $A_{\nu_2\sigma}\mathcal{R}es\ \pi(\nu_1,\sigma)$ for (35) will be

$$\begin{aligned} &A_{\nu_2\sigma}\mathcal{R}es\ \tau_{\nu_1\nu_2} = \tau^{A\nu_2}(\nu_1,\sigma)-\frac{1}{\sigma^2}\left[-0.2576+0.01448\nu_1-0.0019\nu_1^2\right] \\ &+\frac{1}{\sigma^\omega}A_{\nu_2\sigma}\left[A_{\nu_2\sigma}^{-1}(\tau^{A\nu_2}(\nu_1,\sigma))\partial_{\nu_1}A_{\nu_2\sigma}^{-1}(\tau^{A\nu_2}(\nu_1,\sigma))\right] \\ &+\frac{1}{\sigma^\omega}A_{\nu_2\sigma}\left[\partial_{\nu_1}A_{\nu_2\sigma}^{-1}(\pi^{A\nu_2}(\nu_1,\sigma))\right] \\ &A_{\nu_2\sigma}\mathcal{R}es\ \pi_{\nu_1\nu_2} = \pi^{A\nu_2}(\nu_1,\sigma)-\frac{1}{\sigma^2}\left[-0.01448+0.0038\nu_1-0.00144\nu_1^2\right] \\ &+\frac{1}{\sigma^\omega}A_{\nu_2\sigma}\left[\partial_{\nu_1}\left[A_{\nu_2\sigma}^{-1}(\pi^{A\nu_2}(\nu_1,\sigma))A_{\nu_2\sigma}^{-1}(\tau^{A\nu_2}(\nu_1,\sigma))\right]\right] \\ &+\frac{1}{\sigma^\omega}A_{\nu_2\sigma}\left[\partial_{\nu_1}^3A_{\nu_2\sigma}^{-1}(\tau^{A\nu_2}(\nu_1,\sigma))\right] \end{aligned} \tag{38}$$

with the n-th Aboodh residual maps

$$\begin{aligned} &A_{\nu_2\sigma}\mathcal{R}es_n\ \tau_{\nu_1\nu_2} = \tau_n^{A\nu_2}(\nu_1,\sigma)-\frac{1}{\sigma^2}\left[-0.2576+0.01448\nu_1-0.0019\nu_1^2\right] \\ &+\frac{1}{\sigma^\omega}A_{\nu_2\sigma}\left[A_{\nu_2\sigma}^{-1}(\tau_n^{A\nu_2}(\nu_1,\sigma))\partial_{\nu_1}A_{\nu_2\sigma}^{-1}(\tau_n^{A\nu_2}(\nu_1,\sigma))\right] \\ &+\frac{1}{\sigma^\omega}A_{\nu_2\sigma}\left[\partial_{\nu_1}A_{\nu_2\sigma}^{-1}(\pi_n^{A\nu_2}(\nu_1,\sigma))\right] \\ &A_{\nu_2\sigma}\mathcal{R}es_n\ \pi_{\nu_1\nu_2} = \pi_n^{A\nu_2}(\nu_1,\sigma)-\frac{1}{\sigma^2}\left[-0.01448+0.0038\nu_1-0.00144\nu_1^2\right] \\ &+\frac{1}{\sigma^\omega}A_{\nu_2\sigma}\left[\partial_{\nu_1}^3A_{\nu_2\sigma}^{-1}(\tau_n^{A\nu_2}(\nu_1,\sigma))\right] \end{aligned} \tag{39}$$

for $n=1,2,3,\cdots$. Next find the terms of $\langle u_n\rangle_{n\in\mathbb{N}\cup\{0\}}$ and $\langle v_n\rangle_{n\in\mathbb{N}\cup\{0\}}$ by using the relation

$$\lim_{\sigma\to\infty}\sigma^{n\omega+2}A_{\nu_2\sigma}\mathcal{R}es_n\ \tau_{\nu_1\nu_2}=0 \text{ and } \lim_{\sigma\to\infty}\sigma^{n\omega+2}A_{\nu_2\sigma}\mathcal{R}es_n\ \pi_{\nu_1\nu_2}=0 \quad n=1,2,3,\cdots \tag{40}$$

At $n = 1$, by using (36) we have

$$\tau_1^{A\nu_2}(\nu_1, \sigma) = \frac{1}{\sigma^2}\left[-0.2576 + 0.01448\nu_1 - 0.0019\nu_1^2\right] + \frac{u_1(\nu_1)}{\sigma^{\omega+2}}$$

and

$$\pi_1^{A\nu_2}(\nu_1, \sigma) = \frac{1}{\sigma^2}\left[-0.01448 + 0.0038\nu_1 - 0.00144\nu_1^2\right] + \frac{v_1(\nu_1)}{\sigma^{\omega+2}}.$$

From (39) we have

$$\begin{aligned}
&A_{\nu_2\sigma}\mathcal{R}es_1\ \tau_{\nu_1\nu_2} = \frac{u_1(\nu_1)}{\sigma^{\omega+2}} + \frac{1}{\sigma^{\omega}}A_{\nu_2\sigma}\left[A_{\nu_2\sigma}^{-1}(\tau_1^{A\nu_2}(\nu_1,\sigma))\partial_{\nu_1}A_{\nu_2\sigma}^{-1}(\tau_1^{A\nu_2}(\nu_1,\sigma))\right] \\
&+ \frac{1}{\sigma^{\omega}}A_{\nu_2\sigma}\left[\partial_{\nu_1}A_{\nu_2\sigma}^{-1}(\pi_1^{A\nu_2}(\nu_1,\sigma))\right] \\
&A_{\nu_2\sigma}\mathcal{R}es_1\ \pi_{\nu_1\nu_2} = \frac{v_1(\nu_1)}{\sigma^{\omega+2}} + \frac{1}{\sigma^{\omega}}A_{\nu_2\sigma}\left[\partial_{\nu_1}\left[A_{\nu_2\sigma}^{-1}(\pi_1^{A\nu_2}(\nu_1,\sigma))A_{\nu_2\sigma}^{-1}(\tau_1^{A\nu_2}(\nu_1,\sigma))\right]\right] \\
&+ \frac{1}{\sigma^{\omega}}A_{\nu_2\sigma}\left[\partial_{\nu_1}^3 A_{\nu_2\sigma}^{-1}(\tau_1^{A\nu_2}(\nu_1,\sigma))\right]
\end{aligned}$$

This implies

$$\begin{aligned}
&A_{\nu_2\sigma}\mathcal{R}es_1\ \tau_{\nu_1\nu_2} = \frac{1}{\sigma^{\omega+2}}\Big[u_1(\nu_1) - 0.0011862 - 0.00169376\nu_1 - 0.00008208\nu_1^2 \\
&+ 0.00000722\nu_1^3\Big] \\
&+ \frac{1}{\sigma^{2\omega+2}}\Big[(0.0144 - 0.0038\nu_1)u_1(\nu_1) + (0.2576 + 0.0144\nu_1 - 0.0019\nu_1^2)\partial_{\nu_1}u_1(\nu_1) \\
&+ \partial_{\nu_1}v_1(\nu_1)\Big] + \frac{1}{\sigma^{3\omega+2}}\left[\frac{u_1(\nu_1)\partial_{\nu_1}u_1(\nu_1)\Gamma(2\omega+1)}{\Gamma(\omega+1)^2}\right] \\
&A_{\nu_2\sigma}\mathcal{R}es_1\ \pi_{\nu_1\nu_2} = \frac{1}{\sigma^{\omega+2}}\Big[v_1(\nu_1) - 0.001187392 - 0.000906352\nu_1 - 0.000083868\nu_1^2 \\
&+ 0.000010944\nu_1^3\Big] \\
&+ \frac{1}{\sigma^{2\omega+2}}\Big[(-0.2576 + 0.01448\nu_1 - 0.0019\nu_1^2)\partial_{\nu_1}v_1(\nu_1) + \Big(- 0.01448 + 0.0038\nu_1 \\
&- 0.00144\nu_1^2\Big)\partial_{\nu_1}u_1(\nu_1) + \partial_{\nu_1}^3 v_1(\nu_1)\Big] + \frac{1}{\sigma^{3\omega+2}}\left[\frac{\partial_{\nu_1}(u_1(\nu_1)v_1(\nu_1))\Gamma(2\omega+1)}{\Gamma(\omega+1)^2}\right]
\end{aligned} \tag{41}$$

Multiple both sides of (41) by $\sigma^{\omega+2}$ and by (40) we have

$$\begin{aligned}
u_1(\nu_1) &= 0.0011862 + 0.00169376\nu_1 + 0.00008208\nu_1^2 - 0.00000722\nu_1^3 \\
v_1(\nu_1) &= 0.001187392 + 0.000906352\nu_1 + 0.000083868\nu_1^2 - 0.000010944\nu_1^3.
\end{aligned} \tag{42}$$

At $n = 2$, by (36) we have

$$\tau_2^{A\nu_2}(\nu_1, \sigma) = \frac{1}{\sigma^2}[-0.2576 + 0.01448\nu_1 - 0.0019\nu_1^2] + \frac{u_1(\nu_1)}{\sigma^{\omega+2}} + \frac{u_2(\nu_1)}{\sigma^{2\omega+2}}$$

and

$$\pi_2^{A\nu_2}(\nu_1, \sigma) = \frac{1}{\sigma^2}[-0.01448 + 0.0038\nu_1 - 0.00144\nu_1^2] + \frac{v_1(\nu_1)}{\sigma^{\omega+2}} + \frac{v_2(\nu_1)}{\sigma^{2\omega+2}}$$

The eq. (39) becomes

$$A_{\nu_2\sigma}\mathcal{R}es_2\ \tau_{\nu_1\nu_2} = \frac{u_1(\nu_1)}{\sigma^{\omega+2}} + \frac{u_2(\nu_1)}{\sigma^{2\omega+2}} + \frac{1}{\sigma^{\omega}}A_{\nu_2\sigma}\left[A^{-1}_{\nu_2\sigma}(\tau_2^{A\nu_2}(\nu_1,\sigma))\partial_{\nu_1}A^{-1}_{\nu_2\sigma}(\tau_2^{A\nu_2}(\nu_1,\sigma))\right]$$
$$+\frac{1}{\sigma^{\omega}}A_{\nu_2\sigma}\left[\partial_{\nu_1}A^{-1}_{\nu_2\sigma}(\pi_2^{A\nu_2}(\nu_1,\sigma))\right]$$
$$A_{\nu_2\sigma}\mathcal{R}es_2\ \pi_{\nu_1\nu_2} = \frac{v_1(\nu_1)}{\sigma^{\omega+2}} + \frac{v_2(\nu_1)}{\sigma^{2\omega+2}} + \frac{1}{\sigma^{\omega}}A_{\nu_2\sigma}\left[\partial_{\nu_1}\left[A^{-1}_{\nu_2\sigma}(\pi_2^{A\nu_2}(\nu_1,\sigma))A^{-1}_{\nu_2\sigma}(\tau_2^{A\nu_2}(\nu_1,\sigma))\right]\right]$$
$$+\frac{1}{\sigma^{\omega}}A_{\nu_2\sigma}\left[\partial^3_{\nu_1}A^{-1}_{\nu_2\sigma}(\tau_2^{A\nu_2}(\nu_1,\sigma))\right]$$

Similarly, we have

$$\begin{aligned} &u_2(\nu_1) = -1.3718\times 10^{-8}x^4 + 2.09091\times 10^{-7}x^3 - 6.15428\times 10^{-6}x^2\\ &-0.000137736x + 0.001346352\\ &v_2(\nu_1) = -1.55952\times 10^{-7}x^4 + 1.85379\times 10^{-6}x^3 - 8.6685\times 10^{-7}x^2\\ &+0.00001x + 0.000219056. \end{aligned} \tag{43}$$

Put the values of $\langle u_n\rangle_{n\in\mathbb{N}\cup\{0\}}$ and $\langle v_n\rangle_{n\in\mathbb{N}\cup\{0\}}$ in (31) to obtain

$$\begin{aligned} &\tau^{A\nu_2}(\nu_1,\sigma) = \sum_{j=0}^{\infty}\frac{u_j(\nu_1)}{\sigma^{j\omega+2}} = \frac{1}{\sigma^2}u_0(\nu_1) + \frac{1}{\sigma^{\omega+2}}u_1(\nu_1) + \frac{1}{\sigma^{2\omega+2}}u_2(\nu_1) + \cdots\\ &= \frac{1}{\sigma^2}[-0.2576 + 0.01448\nu_1 - 0.0019\nu_1^2]\\ &+\frac{1}{\sigma^{\omega+2}}\Big\{-0.0011862 + 0.00169376\nu_1 + 0.00008208\nu_1^2 - 0.00000722\nu_1^3\Big\}\\ &+\frac{1}{\sigma^{2\omega+2}}\Big[-1.3718\times 10^{-8}x^4 + 2.09091\times 10^{-7}x^3\\ &-6.15428\times 10^{-6}x^2 - 0.000137736x + 0.001346352\Big] + \cdots \end{aligned} \tag{44}$$

and

$$\begin{aligned} &\pi^{A\nu_2}(\nu_1,\sigma) = \sum_{j=0}^{\infty}\frac{v_j(\nu_1)}{\sigma^{j\omega+2}} = \frac{1}{\sigma^2}v_0(\nu_1) + \frac{1}{\sigma^{\omega+2}}v_1(\nu_1) + \frac{1}{\sigma^{2\omega+2}}v_2(\nu_1) + \cdots\\ &= \frac{1}{\sigma^2}[-0.01448 + 0.0038\nu_1 - 0.00144\nu_1^2]\\ &+\frac{1}{\sigma^{\omega+2}}\Big\{0.001187392 + 0.000906352\nu_1 + 0.000083868\nu_1^2 - 0.000010944\nu_1^3\Big\}\\ &+\frac{1}{\sigma^{2\omega+2}}\Big[-1.55952\times 10^{-7}x^4 + 1.85379\times 10^{-6}x^3 - 8.6685\times 10^{-7}x^2\\ &+0.00001x + 0.000219056.\Big] + \cdots \end{aligned} \tag{45}$$

Take the inverse Aboodh transform of (44) and (45),

$$\begin{aligned} &\tau(\nu_1,\nu_2) = -0.2576 + 0.01448\nu_1 - 0.0019\nu_1^2\\ &+\frac{\nu_2^{\omega}}{\Gamma(\omega+1)}\Big\{-0.0011862 + 0.00169376\nu_1 + 0.00008208\nu_1^2 - 0.00000722\nu_1^3\Big\}\\ &+\frac{\nu_2^{2\omega}}{\Gamma(2\omega+1)}\Big[-1.3718\times 10^{-8}x^4 + 2.09091\times 10^{-7}x^3 - 6.15428\times 10^{-6}x^2\\ &-0.000137736x + 0.001346352\Big] + \cdots \end{aligned} \tag{46}$$

and

$$\begin{aligned}\pi(\nu_1,\nu_2) &= -0.01448 + 0.0038\nu_1 - 0.00144\nu_1^2 \\ &+ \frac{\nu_2^{\omega}}{\Gamma(\omega+1)}\Big\{0.001187392 + 0.000906352\nu_1 + 0.000083868\nu_1^2 - 0.000010944\nu_1^3\Big\} \\ &+ \frac{\nu_2^{2\omega}}{\Gamma(2\omega+1)}\Big[-1.55952\times 10^{-7}x^4 + 1.85379\times 10^{-6}x^3 - 8.6685\times 10^{-7}x^2 \\ &+ 0.00001x + 0.000219056.\Big] + \cdots\end{aligned} \tag{47}$$

Here we will find the analytical solution of the Caputo WBKEs (2) with the exact initial condition. Consider the following Caputo WBKEs

$$\begin{cases} {}^{c}\mathcal{D}_{\nu_2}^{\omega}\tau(\nu_1,\nu_2) + \tau(\nu_1,\nu_2)\partial_{\nu_1}\tau(\nu_1,\nu_2) + \partial_{\nu_1}\pi(\nu_1,\nu_2) = 0 \\ {}^{c}\mathcal{D}_{\nu_2}^{\omega}\pi(\nu_1,\nu_2) + \partial_{\nu_1}\left[\tau(\nu_1,\nu_2)\pi(\nu_1,\nu_2)\right] + \partial_{\nu_1}^3\tau(\nu_1,\nu_2) = 0 \\ \tau(\nu_1,0) = 0.005 - 0.2\coth(0.1\nu_1+1) \\ \pi(\nu_1,0) = -0.02\ \mathrm{csch}(0.1\nu_1+1). \end{cases} \tag{48}$$

Recall [36] that if $\omega = 1$ then the exact solution of (48) is

$$\begin{aligned}\tau(\nu_1,0) &= 0.005 - 0.2\coth(0.1\nu_1 - 0.0005\nu_2 + 1) \\ \pi(\nu_1,0) &= -0.02\ \mathrm{csch}(0.1\nu_1 - 0.0005\nu_2 + 1).\end{aligned} \tag{49}$$

Use Aboodh transforms on (48) to obtain

$$\begin{aligned}\tau^{A\nu_2}(\nu_1,\sigma) &= \frac{1}{\sigma^2}\left[0.005 - 0.2\coth(0.1\nu_1+1)\right] \\ &- \frac{1}{\sigma^{\omega}}A_{\nu_2\sigma}\left[A^{-1}_{\nu_2\sigma}(\tau^{A\nu_2}(\nu_1,\sigma))\partial_{\nu_1}A^{-1}_{\nu_2\sigma}(\tau^{A\nu_2}(\nu_1,\sigma))\right] - \frac{1}{\sigma^{\omega}}A_{\nu_2\sigma}\left[\partial_{\nu_1}A^{-1}_{\nu_2\sigma}(\pi^{A\nu_2}(\nu_1,\sigma))\right] \\ \pi^{A\nu_2}(\nu_1,\sigma) &= \frac{1}{\sigma^2}\left[-0.02\ \mathrm{csch}(0.1\nu_1+1)\right] \\ &- \frac{1}{\sigma^{\omega}}A_{\nu_2\sigma}\left[\partial_{\nu_1}\left[A^{-1}_{\nu_2\sigma}(\pi^{A\nu_2}(\nu_1,\sigma))A^{-1}_{\nu_2\sigma}(\tau^{A\nu_2}(\nu_1,\sigma))\right]\right] - \frac{1}{\sigma^{\omega}}A_{\nu_2\sigma}\left[\partial^3_{\nu_1}A^{-1}_{\nu_2\sigma}(\tau^{A\nu_2}(\nu_1,\sigma))\right].\end{aligned} \tag{50}$$

The partial sums sequences $\left\langle\tau_n^{A\nu_2}\right\rangle_{n\in\mathbb{N}\cup\{0\}}$ and $\left\langle\pi_n^{A\nu_2}\right\rangle_{n\in\mathbb{N}\cup\{0\}}$ are given by

$$\begin{aligned}\tau_n^{A\nu_2}(\nu_1,\sigma) &= \frac{1}{\sigma^2}\left[0.005 - 0.2\coth(0.1\nu_1+1)\right] + \sum_{j=1}^{n}\frac{u_j(\nu_1)}{\sigma^{j\omega+2}} \\ \pi_n^{A\nu_2}(\nu_1,\sigma) &= \frac{1}{\sigma^2}\left[-0.02\ \mathrm{csch}(0.1\nu_1+1)\right] + \sum_{j=1}^{n}\frac{v_j(\nu_1)}{\sigma^{j\omega+2}}.\end{aligned} \tag{51}$$

The n-th constructions of (50) are given by

$$\begin{aligned}\tau_n^{A\nu_2}(\nu_1,\sigma) &= \frac{1}{\sigma^2}\left[0.005 - 0.2\coth(0.1\nu_1+1)\right] \\ &- \frac{1}{\sigma^{\omega}}A_{\nu_2\sigma}\left[A^{-1}_{\nu_2\sigma}(\tau_n^{A\nu_2}(\nu_1,\sigma))\partial_{\nu_1}A^{-1}_{\nu_2\sigma}(\tau_n^{A\nu_2}(\nu_1,\sigma))\right] - \frac{1}{\sigma^{\omega}}A_{\nu_2\sigma}\left[\partial_{\nu_1}A^{-1}_{\nu_2\sigma}(\pi_n^{A\nu_2}(\nu_1,\sigma))\right] \\ \pi_n^{A\nu_2}(\nu_1,\sigma) &= \frac{1}{\sigma^2}\left[-0.02\ \mathrm{csch}(0.1\nu_1+1)\right] \\ &- \frac{1}{\sigma^{\omega}}A_{\nu_2\sigma}\left[\partial_{\nu_1}\left[A^{-1}_{\nu_2\sigma}(\pi_n^{A\nu_2}(\nu_1,\sigma))A^{-1}_{\nu_2\sigma}(\tau_n^{A\nu_2}(\nu_1,\sigma))\right]\right] \\ &- \frac{1}{\sigma^{\omega}}A_{\nu_2\sigma}\left[\partial^3_{\nu_1}A^{-1}_{\nu_2\sigma}(\tau_n^{A\nu_2}(\nu_1,\sigma))\right].\end{aligned} \tag{52}$$

for $n = 1, 2, 3, \cdots$. The Aboodh residual maps, $A_{\nu_2\sigma}\mathcal{R}es\ \tau(\nu_1, \sigma)$ and $A_{\nu_2\sigma}\mathcal{R}es\ \pi(\nu_1, \sigma)$ for (50) will be

$$\begin{aligned}
&A_{\nu_2\sigma}\mathcal{R}es\ \tau_{\nu_1\nu_2} = \tau^{A\nu_2}(\nu_1, \sigma) - \frac{1}{\sigma^2}\left[0.005 - 0.2\coth(0.1\nu_1 + 1)\right] \\
&+ \frac{1}{\sigma^\omega}A_{\nu_2\sigma}\left[A^{-1}_{\nu_2\sigma}(\tau^{A\nu_2}(\nu_1, \sigma))\partial_{\nu_1}A^{-1}_{\nu_2\sigma}(\tau^{A\nu_2}(\nu_1, \sigma))\right] \\
&+ \frac{1}{\sigma^\omega}A_{\nu_2\sigma}\left[\partial_{\nu_1}A^{-1}_{\nu_2\sigma}(\pi^{A\nu_2}(\nu_1, \sigma))\right] \\
&A_{\nu_2\sigma}\mathcal{R}es\ \pi_{\nu_1\nu_2} = \pi^{A\nu_2}(\nu_1, \sigma) - \frac{1}{\sigma^2}\left[-0.02\ \mathrm{csch}(0.1\nu_1 + 1)\right] \\
&+ \frac{1}{\sigma^\omega}A_{\nu_2\sigma}\left[\partial_{\nu_1}\left[A^{-1}_{\nu_2\sigma}(\pi^{A\nu_2}(\nu_1, \sigma))A^{-1}_{\nu_2\sigma}(\tau^{A\nu_2}(\nu_1, \sigma))\right]\right] \\
&+ \frac{1}{\sigma^\omega}A_{\nu_2\sigma}\left[\partial^3_{\nu_1}A^{-1}_{\nu_2\sigma}(\tau^{A\nu_2}(\nu_1, \sigma))\right]
\end{aligned} \tag{53}$$

with the n-th residual maps

$$\begin{aligned}
&A_{\nu_2\sigma}\mathcal{R}es_n\ \tau_{\nu_1\nu_2} = \tau_n^{A\nu_2}(\nu_1, \sigma) - \frac{1}{\sigma^2}\left[0.005 - 0.2\coth(0.1\nu_1 + 1)\right] \\
&+ \frac{1}{\sigma^\omega}A_{\nu_2\sigma}\left[A^{-1}_{\nu_2\sigma}(\tau_n^{A\nu_2}(\nu_1, \sigma))\partial_{\nu_1}A^{-1}_{\nu_2\sigma}(\tau_n^{A\nu_2}(\nu_1, \sigma))\right] \\
&+ \frac{1}{\sigma^\omega}A_{\nu_2\sigma}\left[\partial_{\nu_1}A^{-1}_{\nu_2\sigma}(\pi_n^{A\nu_2}(\nu_1, \sigma))\right] \\
&A_{\nu_2\sigma}\mathcal{R}es_n\ \pi_{\nu_1\nu_2} = \pi_n^{A\nu_2}(\nu_1, \sigma) - \frac{1}{\sigma^2}\left[-0.02\ \mathrm{csch}(0.1\nu_1 + 1)\right] \\
&+ \frac{1}{\sigma^\omega}A_{\nu_2\sigma}\left[\partial^3_{\nu_1}A^{-1}_{\nu_2\sigma}(\tau_n^{A\nu_2}(\nu_1, \sigma))\right]
\end{aligned} \tag{54}$$

for $n = 1, 2, 3, \cdots$. Find the terms of $\langle u_n\rangle_{n\in\mathbb{N}\cup\{0\}}$ and $\langle v_n\rangle_{n\in\mathbb{N}\cup\{0\}}$ by using the following

$$\lim_{\sigma\to\infty}\sigma^{n\omega+2}A_{\nu_2\sigma}\mathcal{R}es_n\ \tau_{\nu_1\nu_2} = 0 \text{ and } \lim_{\sigma\to\infty}\sigma^{n\omega+2}A_{\nu_2\sigma}\mathcal{R}es_n\ \pi_{\nu_1\nu_2} = 0 \quad n = 1, 2, 3, \cdots \tag{55}$$

At $n = 1$, by using (51) we have

$$\tau_1^{A\nu_2}(\nu_1, \sigma) = \frac{1}{\sigma^2}\left[0.005 - 0.2\coth(0.1\nu_1 + 1)\right] + \frac{u_1(\nu_1)}{\sigma^{\omega+2}}$$

and

$$\pi_1^{A\nu_2}(\nu_1, \sigma) = \frac{1}{\sigma^2}\left[-0.02\ \mathrm{csch}(0.1\nu_1 + 1)\right] + \frac{v_1(\nu_1)}{\sigma^{\omega+2}}.$$

Also (54) becomes

$$\begin{aligned}
&A_{\nu_2\sigma}\mathcal{R}es_1\ \tau_{\nu_1\nu_2} = \frac{u_1(\nu_1)}{\sigma^{\omega+2}} + \frac{1}{\sigma^\omega}A_{\nu_2\sigma}\left[A^{-1}_{\nu_2\sigma}(\tau_1^{A\nu_2}(\nu_1, \sigma))\partial_{\nu_1}A^{-1}_{\nu_2\sigma}(\tau_1^{A\nu_2}(\nu_1, \sigma))\right] \\
&+ \frac{1}{\sigma^\omega}A_{\nu_2\sigma}\left[\partial_{\nu_1}A^{-1}_{\nu_2\sigma}(\pi_1^{A\nu_2}(\nu_1, \sigma))\right] \\
&A_{\nu_2\sigma}\mathcal{R}es_1\ \pi_{\nu_1\nu_2} = \frac{v_1(\nu_1)}{\sigma^{\omega+2}} + \frac{1}{\sigma^\omega}A_{\nu_2\sigma}\left[\partial_{\nu_1}\left[A^{-1}_{\nu_2\sigma}(\pi_1^{A\nu_2}(\nu_1, \sigma))A^{-1}_{\nu_2\sigma}(\tau_1^{A\nu_2}(\nu_1, \sigma))\right]\right] \\
&+ \frac{1}{\sigma^\omega}A_{\nu_2\sigma}\left[\partial^3_{\nu_1}A^{-1}_{\nu_2\sigma}(\tau_1^{A\nu_2}(\nu_1, \sigma))\right]
\end{aligned}$$

This implies

$$\begin{aligned}
&A_{\nu_2\sigma}\mathcal{R}es_1\ \tau_{\nu_1\nu_2} = \frac{1}{\sigma^{\omega+2}}\Big[u_1(\nu_1) + 0.0001\ \operatorname{csch}^2(0.1\nu_1+1)\\
&-0.004\coth(0.1\nu_1+1)\ \operatorname{csch}^2(0.1\nu_1+1) + 0.002\coth(0.1\nu_1+1)\ \operatorname{csch}(0.1\nu_1+1)\Big]\\
&\frac{1}{\sigma^{2\omega+2}}\Big[0.02\ \operatorname{csch}^2(0.1\nu_1+1)u_1(\nu_1) + (0.005-0.2\coth(0.1\nu_1+1))\partial u_1(\nu_1)\\
&+\partial v_1(\nu_1)\Big] + \frac{1}{\sigma^{3\omega+2}}\left[\frac{u_1(\nu_1)\partial_{\nu_1}u_1(\nu_1)\Gamma(2\omega+1)}{\Gamma(\omega+1)^2}\right]\\
&A_{\nu_2\sigma}\mathcal{R}es_1\ \pi_{\nu_1\nu_2} = \frac{1}{\sigma^{\omega+2}}\Big[v_1(\nu_1) - 0.0004\ \operatorname{csch}^3(0.1\nu_1+1))\\
&+0.002\coth(0.1\nu_1+1))\ \operatorname{csch}(0.1\nu_1+1))(0.005-0.2\coth(0.1\nu_1+1))\\
&-0.0008\ \operatorname{csch}^2(0.1\nu_1+1))\coth^2(0.1\nu_1+1)) + 0.0004\ \operatorname{csch}^4(0.1\nu_1+1))\Big]\\
&+\frac{1}{\sigma^{2\omega+2}}\Big[\partial_{\nu_1}[(0.005-0.2\coth(0.1\nu_1+1)))v_1(\nu_1)]\\
&-\partial_{\nu_1}[0.02\ \operatorname{csch}(0.1\nu_1+1))u_1(\nu_1) + \partial^3_{\nu_1}u_1(\nu_1)\Big]\\
&+\frac{1}{\sigma^{3\omega+2}}\left[\frac{\partial_{\nu_1}(u_1(\nu_1)v_1(\nu_1))\Gamma(2\omega+1)}{\Gamma(\omega+1)^2}\right]
\end{aligned}\tag{56}$$

By multiplying both sides of (56) by $\sigma^{\omega+2}$ and by (55) we have

$$\begin{aligned}
&u_1(\nu_1) = 0.004\coth(0.1\nu_1+1)\ \operatorname{csch}^2(0.1\nu_1+1) - 0.0001\ \operatorname{csch}^2(0.1\nu_1+1)\\
&-0.002\coth(0.1\nu_1+1)\ \operatorname{csch}(0.1\nu_1+1)\\
&v_1(\nu_1) = 0.0004\ \operatorname{csch}^3(0.1\nu_1+1)\\
&-0.002\coth(0.1\nu_1+1))\ \operatorname{csch}(0.1\nu_1+1))(0.005-0.2\coth(0.1\nu_1+1))\\
&+0.0008\ \operatorname{csch}^2(0.1\nu_1+1))\coth^2(0.1\nu_1+1)) - 0.0004\ \operatorname{csch}^4(0.1\nu_1+1)).
\end{aligned}\tag{57}$$

At $n=2$, by (51) we have

$$\tau_2^{A\nu_2}(\nu_1,\sigma) = \frac{1}{\sigma^2}[0.005-0.2\coth(0.1\nu_1+1)] + \frac{u_1(\nu_1)}{\sigma^{\omega+2}} + \frac{u_2(\nu_1)}{\sigma^{2\omega+2}}$$

and

$$\pi_2^{A\nu_2}(\nu_1,\sigma) = \frac{1}{\sigma^2}[-0.02\ \operatorname{csch}(0.1\nu_1+1)] + \frac{v_1(\nu_1)}{\sigma^{\omega+2}} + \frac{v_2(\nu_1)}{\sigma^{2\omega+2}}.$$

Eq (54) becomes

$$\begin{aligned}
&A_{\nu_2\sigma}\mathcal{R}es_2\ \tau_{\nu_1\nu_2} = \frac{u_1(\nu_1)}{\sigma^{\omega+2}} + \frac{u_2(\nu_1)}{\sigma^{2\omega+2}} + \frac{1}{\sigma^\omega}A_{\nu_2\sigma}\left[A^{-1}_{\nu_2\sigma}(\tau_2^{A\nu_2}(\nu_1,\sigma))\partial_{\nu_1}A^{-1}_{\nu_2\sigma}(\tau_2^{A\nu_2}(\nu_1,\sigma))\right]\\
&+\frac{1}{\sigma^\omega}A_{\nu_2\sigma}\left[\partial_{\nu_1}A^{-1}_{\nu_2\sigma}(\pi_2^{A\nu_2}(\nu_1,\sigma))\right]\\
&A_{\nu_2\sigma}\mathcal{R}es_2\ \pi_{\nu_1\nu_2} = \frac{v_1(\nu_1)}{\sigma^{\omega+2}} + \frac{v_2(\nu_1)}{\sigma^{2\omega+2}} + \frac{1}{\sigma^\omega}A_{\nu_2\sigma}\left[\partial_{\nu_1}\left[A^{-1}_{\nu_2\sigma}(\pi_2^{A\nu_2}(\nu_1,\sigma))A^{-1}_{\nu_2\sigma}(\tau_2^{A\nu_2}(\nu_1,\sigma))\right]\right]\\
&+\frac{1}{\sigma^\omega}A_{\nu_2\sigma}\left[\partial^3_{\nu_1}A^{-1}_{\nu_2\sigma}(\tau_2^{A\nu_2}(\nu_1,\sigma))\right]
\end{aligned}$$

Similarly, we have

$$
\begin{aligned}
u_2(\nu_1) &= -0.0040001 \operatorname{csch}^2(0.1\nu_1+1)\coth(0.1\nu_1+1) + 0.0001 \operatorname{csch}^2(0.1\nu_1+1) \\
&+ 0.002\coth(0.1\nu_1+1) \operatorname{csch}(0.1\nu_1+1) \\
&- 0.000088 \operatorname{csch}^4(0.1\nu_1+1)\coth(0.1\nu_1+1) + 0.0000022 \operatorname{csch}^4(0.1\nu_1+1) \\
&+ 0.000044 \operatorname{csch}^3(0.1\nu_1+1)\coth(0.1\nu_1+1) \\
&- 0.00016\coth^3(0.1\nu_1+1) \operatorname{csch}^2(0.1\nu_1+1) \\
&+ 0.000008\coth^2(0.1\nu_1+1) \operatorname{csch}^2(0.1\nu_1+1) \\
&+ 0.00004\coth^3(0.1\nu_1+1) \operatorname{csch}(0.1\nu_1+1) \\
&- 0.000001 \operatorname{csch}^3(0.1\nu_1+1) - 0.000001\coth^2(0.1\nu_1+1) \operatorname{csch}(0.1\nu_1+1)
\end{aligned} \tag{58}
$$

and

$$
\begin{aligned}
v_2(\nu_1) &= -0.000072 \operatorname{csch}^3(0.1\nu_1+1)\coth^2(0.1\nu_1+1) - 0.000008 \operatorname{csch}^5(0.1\nu_1+1) \\
&+ 0.000002 \operatorname{csch}^3(0.1\nu_1+1)\coth(0.1\nu_1+1) \\
&+ 0.00072 \operatorname{csch}^2(0.1\nu_1+1)\coth^2(0.1\nu_1+1) + 0.000396 \operatorname{csch}^4(0.1\nu_1+1) \\
&+ 0.0002 \operatorname{csch}^3(0.1\nu_1+1) + 0.0002\coth^2(0.1\nu_1+1) \operatorname{csch}(0.1\nu_1+1) \\
&- 0.0000004\coth^3(0.1\nu_1+1) \operatorname{csch}(0.1\nu_1+1) \\
&- 0.0000008\coth^3(0.1\nu_1+1) \operatorname{csch}^2(0.1\nu_1+1) \\
&- 0.000008 \operatorname{csch}^5(0.1\nu_1+1) - 0.000008\coth^4(0.1\nu_1+1) \operatorname{csch}(0.1\nu_1+1) \\
&- 0.000016 \operatorname{csch}^4(0.1\nu_1+1)\coth^2(0.1\nu_1+1) \\
&- 0.000032\coth^4(0.1\nu_1+1) \operatorname{csch}^2(0.1\nu_1+1) + 0.000008 \operatorname{csch}^6(0.1\nu_1+1)) \\
&+ 0.00002 \operatorname{csch}^2(0.1\nu_1+1)\coth(0.1\nu_1+1)
\end{aligned} \tag{59}
$$

Next put the terms of $\langle u_n\rangle_{n\in\mathbb{N}\cup\{0\}}$ and $\langle v_n\rangle_{n\in\mathbb{N}\cup\{0\}}$ in (31) to get

$$
\begin{aligned}
\tau^{A\nu_2}(\nu_1,\sigma) &= \sum_{j=0}^{\infty}\frac{u_j(\nu_1)}{\sigma^{j\omega+2}} = \frac{1}{\sigma^2}u_0(\nu_1) + \frac{1}{\sigma^{\omega+2}}u_1(\nu_1) + \frac{1}{\sigma^{2\omega+2}}u_2(\nu_1) + \cdots \\
&= \frac{1}{\sigma^2}[0.005 - 0.2\coth(0.1\nu_1+1)] + \frac{1}{\sigma^{\omega+2}}\Big\{0.004\coth(0.1\nu_1+1) \operatorname{csch}^2(0.1\nu_1+1) \\
&- 0.0001 \operatorname{csch}^2(0.1\nu_1+1) - 0.002\coth(0.1\nu_1+1) \operatorname{csch}(0.1\nu_1+1)\Big\} \\
&+ \frac{1}{\sigma^{2\omega+2}}\Big[- 0.0040001 \operatorname{csch}^2(0.1\nu_1+1)\coth(0.1\nu_1+1) + 0.0001 \operatorname{csch}^2(0.1\nu_1+1) \\
&+ 0.002\coth(0.1\nu_1+1) \operatorname{csch}(0.1\nu_1+1) \\
&- 0.000088 \operatorname{csch}^4(0.1\nu_1+1)\coth(0.1\nu_1+1) + 0.0000022 \operatorname{csch}^4(0.1\nu_1+1) \\
&+ 0.000044 \operatorname{csch}^3(0.1\nu_1+1)\coth(0.1\nu_1+1) \\
&- 0.00016\coth^3(0.1\nu_1+1) \operatorname{csch}^2(0.1\nu_1+1) \\
&+ 0.000008\coth^2(0.1\nu_1+1) \operatorname{csch}^2(0.1\nu_1+1) \\
&+ 0.00004\coth^3(0.1\nu_1+1) \operatorname{csch}(0.1\nu_1+1) \\
&- 0.000001 \operatorname{csch}^3(0.1\nu_1+1) - 0.000001\coth^2(0.1\nu_1+1) \operatorname{csch}(0.1\nu_1+1)\Big] + \cdots
\end{aligned} \tag{60}
$$

and

$$\pi^{A\nu_2}(\nu_1,\sigma)=\sum_{j=0}^{\infty}\frac{v_j(\nu_1)}{\sigma^{j\omega+2}}=\frac{1}{\sigma^2}v_0(\nu_1)+\frac{1}{\sigma^{\omega+2}}v_1(\nu_1)+\frac{1}{\sigma^{2\omega+2}}v_2(\nu_1)+\cdots$$
$$=\frac{1}{\sigma^2}[-0.02\ \text{csch}(0.1\nu_1+1)]+\frac{1}{\sigma^{\omega+2}}\Big\{0.0004\ \text{csch}^3(0.1\nu_1+1)$$
$$-0.002\coth(0.1\nu_1+1))\ \text{csch}(0.1\nu_1+1))(0.005-0.2\coth(0.1\nu_1+1))$$
$$+0.0008\ \text{csch}^2(0.1\nu_1+1))\coth^2(0.1\nu_1+1))-0.0004\ \text{csch}^4(0.1\nu_1+1))\Big\}$$
$$+\frac{1}{\sigma^{2\omega+2}}\Big[-0.000072\ \text{csch}^3(0.1\nu_1+1)\coth^2(0.1\nu_1+1)-0.000008\ \text{csch}^5(0.1\nu_1+1)$$
$$+0.000002\ \text{csch}^3(0.1\nu_1+1)\coth(0.1\nu_1+1)$$
$$+0.00072\ \text{csch}^2(0.1\nu_1+1)\coth^2(0.1\nu_1+1)+0.000396\ \text{csch}^4(0.1\nu_1+1)$$
$$+0.0002\ \text{csch}^3(0.1\nu_1+1)+0.0002\coth^2(0.1\nu_1+1)\ \text{csch}(0.1\nu_1+1)$$
$$-0.0000004\coth^3(0.1\nu_1+1)\ \text{csch}(0.1\nu_1+1)$$
$$-0.0000008\coth^3(0.1\nu_1+1)\ \text{csch}^2(0.1\nu_1+1)$$
$$-0.000008\ \text{csch}^5(0.1\nu_1+1)-0.000008\coth^4(0.1\nu_1+1)\ \text{csch}(0.1\nu_1+1)$$
$$-0.000016\ \text{csch}^4(0.1\nu_1+1)\coth^2(0.1\nu_1+1)$$
$$-0.000032\coth^4(0.1\nu_1+1)\ \text{csch}^2(0.1\nu_1+1)+0.000008\ \text{csch}^6(0.1\nu_1+1))$$
$$+0.00002\ \text{csch}^2(0.1\nu_1+1)\coth(0.1\nu_1+1)\Big]+\cdots \tag{61}$$

Use the inverse Aboodh transform of (61) and (60),

$$\tau(\nu_1,\nu_2)=0.005-0.2\coth(0.1\nu_1+1)+\frac{\nu_2^{\omega}}{\Gamma(\omega+1)}\Big\{0.004\coth(0.1\nu_1+1)$$
$$csch^2(0.1\nu_1+1)-0.0001\ \text{csch}^2(0.1\nu_1+1)-0.002\coth(0.1\nu_1+1)\ \text{csch}(0.1\nu_1+1)\Big\}$$
$$+\frac{\nu_2^{2\omega}}{\Gamma(2\omega+1)}\Big[-0.0040001\ \text{csch}^2(0.1\nu_1+1)\coth(0.1\nu_1+1)+0.0001\ \text{csch}^2(0.1\nu_1+1)$$
$$+0.002\coth(0.1\nu_1+1)\ \text{csch}(0.1\nu_1+1)$$
$$-0.000088\ \text{csch}^4(0.1\nu_1+1)\coth(0.1\nu_1+1)+0.0000022\ \text{csch}^4(0.1\nu_1+1)$$
$$+0.000044\ \text{csch}^3(0.1\nu_1+1)\coth(0.1\nu_1+1)$$
$$-0.00016\coth^3(0.1\nu_1+1)\ \text{csch}^2(0.1\nu_1+1)$$
$$+0.000008\coth^2(0.1\nu_1+1)\ \text{csch}^2(0.1\nu_1+1)$$
$$+0.00004\coth^3(0.1\nu_1+1)\ \text{csch}(0.1\nu_1+1)$$
$$-0.000001\ \text{csch}^3(0.1\nu_1+1)-0.000001\coth^2(0.1\nu_1+1)\ \text{csch}(0.1\nu_1+1)\Big]+\cdots \tag{62}$$

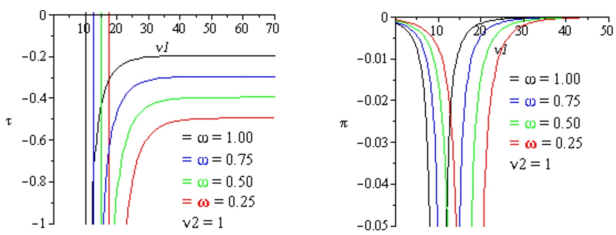

Figure 1: ARPST graphs for solutions (62) and (63) with $\nu_2 = 1$ and some values of ν_1 and ω.

and

$$\begin{aligned}
\pi(\nu_1,\nu_2) &= -0.02\ \operatorname{csch}(0.1\nu_1+1) + \frac{\nu_2^{\omega}}{\Gamma(\omega+1)}\Big\{0.0004\ \operatorname{csch}^3(0.1\nu_1+1) \\
&- 0.002\coth(0.1\nu_1+1))\ \operatorname{csch}(0.1\nu_1+1))(0.005-0.2\coth(0.1\nu_1+1)) \\
&+ 0.0008\ \operatorname{csch}^2(0.1\nu_1+1))\coth^2(0.1\nu_1+1)) - 0.0004\ \operatorname{csch}^4(0.1\nu_1+1))\Big\} \\
&+ \frac{\nu_2^{2\omega}}{\Gamma(2\omega+1)}\Big[-0.000072\ \operatorname{csch}^3(0.1\nu_1+1)\coth^2(0.1\nu_1+1) \\
&- 0.000008\ \operatorname{csch}^5(0.1\nu_1+1) + 0.000002\ \operatorname{csch}^3(0.1\nu_1+1)\coth(0.1\nu_1+1) \\
&+ 0.00072\ \operatorname{csch}^2(0.1\nu_1+1)\coth^2(0.1\nu_1+1) + 0.000396\ \operatorname{csch}^4(0.1\nu_1+1) \\
&+ 0.0002\ \operatorname{csch}^3(0.1\nu_1+1) + 0.0002\coth^2(0.1\nu_1+1)\ \operatorname{csch}(0.1\nu_1+1) \\
&- 0.0000004\coth^3(0.1\nu_1+1)\ \operatorname{csch}(0.1\nu_1+1) \\
&- 0.0000008\coth^3(0.1\nu_1+1)\ \operatorname{csch}^2(0.1\nu_1+1) \\
&- 0.000008\ \operatorname{csch}^5(0.1\nu_1+1) - 0.000008\coth^4(0.1\nu_1+1)\ \operatorname{csch}(0.1\nu_1+1) \\
&- 0.000016\ \operatorname{csch}^4(0.1\nu_1+1)\coth^2(0.1\nu_1+1) \\
&- 0.000032\coth^4(0.1\nu_1+1)\ \operatorname{csch}^2(0.1\nu_1+1) + 0.000008\ \operatorname{csch}^6(0.1\nu_1+1)) \\
&+ 0.00002\ \operatorname{csch}^2(0.1\nu_1+1)\coth(0.1\nu_1+1)\Big] + \cdots
\end{aligned} \tag{63}$$

Table 1: Comparing absolute errors of ARPST solution $\tau(\nu_1,\nu_2)$ at $\omega = 1$, with ADM [34], OHAM [35] and SDM [36] in Caputo WBKEs (34).

ν_1	ν_2	ARPST	ADM [34]	OHAM [35]	SDM [36]
0.1	0.1	3.01653×10^{-12}	8.16297×10^{-7}	6.35267×10^{-5}	5.55112×10^{-17}
	0.3	2.01534×10^{-10}	7.64245×10^{-7}	1.90854×10^{-4}	5.55112×10^{-17}
	0.5	3.30171×10^{-11}	7.16083×10^{-7}	3.18548×10^{-4}	6.55134×10^{-16}
0.2	0.1	4.00331×10^{-9}	3.26243×10^{-6}	6.18931×10^{-5}	1.11022×10^{-16}
	0.3	2.30154×10^{-13}	3.05458×10^{-6}	1.85945×10^{-4}	1.11022×10^{-16}
	0.5	4.12461×10^{-12}	2.86226×10^{-6}	3.10352×10^{-4}	7.77156×10^{-16}
0.3	0.1	5.65473×10^{-11}	7.33445×10^{-6}	6.03098×10^{-5}	0
	0.3	7.39784×10^{-13}	6.86758×10^{-6}	1.81187×10^{-4}	1.11023×10^{-16}
	0.5	1.39861×10^{-10}	6.43557×10^{-6}	3.02408×10^{-4}	6.66133×10^{-16}
0.4	0.1	4.29713×10^{-9}	1.30286×10^{-5}	5.87749×10^{-5}	0
	0.3	3.30984×10^{-12}	1.22000×10^{-5}	1.76574×10^{-4}	5.55112×10^{-17}
	0.5	1.35641×10^{-11}	1.14333×10^{-5}	2.94708×10^{-4}	6.10623×10^{-16}
0.5	0.1	5.29813×10^{-10}	2.03415×10^{-5}	5.72865×10^{-4}	0
	0.3	9.38964×10^{-8}	1.90489×10^{-5}	1.72102×10^{-4}	1.11020×10^{-16}
	0.5	3.87681×10^{-11}	1.78528×10^{-5}	2.87240×10^{-4}	6.10623×10^{-16}

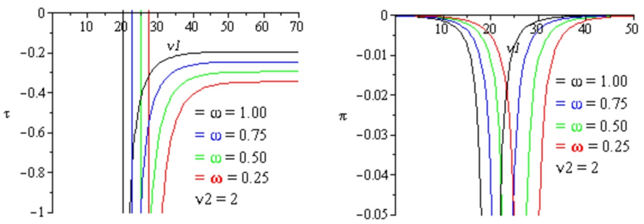

Figure 2: ARPST graphs for solutions (62) and (63) with $\nu_2 = 2$.

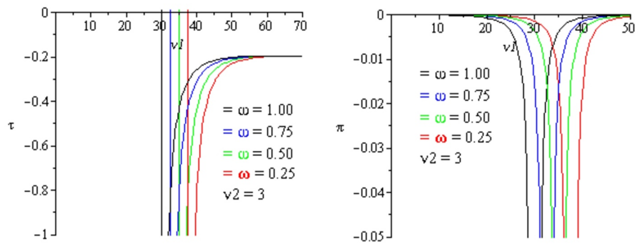

Figure 3: ARPST graphs for solutions (62) and (63) with $\nu_2 = 3$.

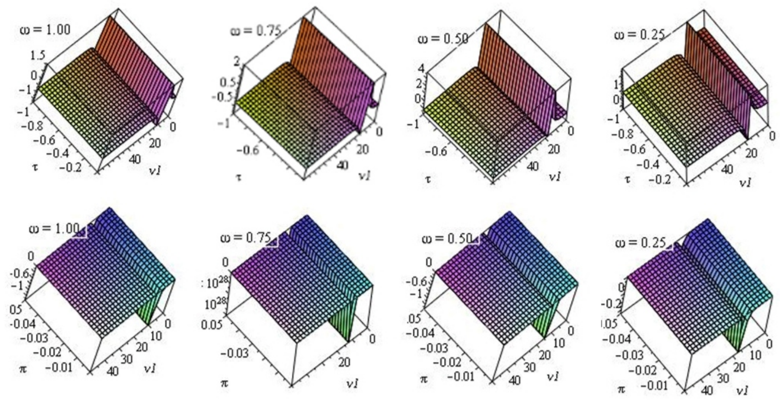

Figure 4: Surface plot of solution (62) and (63) with $\nu_2 = 1$.

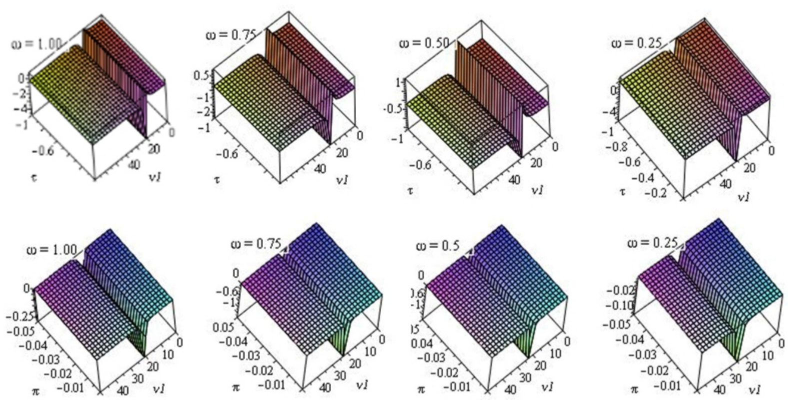

Figure 5: Surface plot of solution (62) and (63) with $\nu_2 = 2$.

Table 2: Comparing absolute errors of ARPST solution $\pi(\nu_1, \nu_2)$ at $\omega = 1$, with ADM [34], OHAM [35] and SDM [36] in Caputo WBKEs (34).

ν_1	ν_2	ARPST	ADM [34]	OHAM [35]	SDM [36]
0.1	0.1	3.14371×10^{-14}	5.88676×10^{-5}	1.65942×10^{-5}	3.46945×10^{-18}
	0.3	4.21892×10^{-16}	5.56914×10^{-5}	4.98691×10^{-5}	5.20417×10^{-17}
	0.5	1.34711×10^{-15}	5.27169×10^{-5}	8.26491×10^{-4}	3.60822×10^{-16}
0.2	0.1	9.08933×10^{-13}	1.18213×10^{-4}	1.06812×10^{-5}	6.93889×10^{-18}
	0.3	8.26574×10^{-12}	1.11833×10^{-4}	4.83269×10^{-5}	4.68375×10^{-17}
	0.5	1.28756×10^{-11}	1.05858×10^{-4}	7.94290×10^{-4}	3.46945×10^{-16}
0.3	0.1	4.28943×10^{-14}	1.78041×10^{-4}	1.55880×10^{-5}	6.93889×10^{-18}
	0.3	3.10897×10^{-11}	1.68429×10^{-4}	4.68439×10^{-5}	3.98986×10^{-17}
	0.5	6.30101×10^{-13}	1.59428×10^{-4}	7.63646×10^{-4}	3.22659×10^{-16}
0.4	0.1	4.20243×10^{-15}	2.38356×10^{-4}	1.51135×10^{-5}	5.20417×10^{-18}
	0.3	3.30384×10^{-12}	2.25483×10^{-4}	4.54174×10^{-5}	3.46945×10^{-17}
	0.5	7.21901×10^{-14}	2.13430×10^{-4}	7.34471×10^{-4}	3.05311×10^{-16}
0.5	0.1	6.35163×10^{-12}	2.99162×10^{-4}	1.46569×10^{-5}	1.73472×10^{-18}
	0.3	3.28964×10^{-13}	2.83001×10^{-4}	4.40448×10^{-5}	4.16334×10^{-17}
	0.5	5.10971×10^{-11}	2.67868×10^{-4}	7.06678×10^{-4}	2.87964×10^{-16}

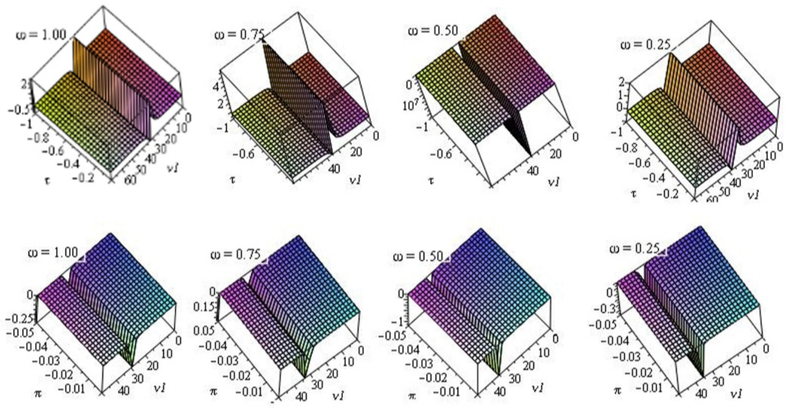

Figure 6: Surface plot of solution (62) and (63) with $\nu_2 = 3$.

5 Numerical discussion

In this section, we will introduce the solutions (62) and (63) of Caputo WBKEs (34) as numerically and graphically by giving some fractional-order values ω in Figures (1)-(5) and in Tables (1), (2). Table (1) presents the dynamical behaviour of the analytical solutions of Caputo WBKEs (34) where involves some comparisons fro the absolute error of the ARPST solution $\tau(\nu_1, \nu_2)$ and the exact solution at $\omega = 1$ with the obtained solutions of Caputo WBKEs (34) by ADM [34], OHAM [35] and SDM [36]. Table (2) presents the dynamical behaviour of the analytical solutions of Caputo WBKEs (34) where involves some comparisons fro the absolute error of the ARPST solution $\pi(\nu_1, \nu_2)$ and the exact solution at $\omega = 1$ with the obtained solutions of Caputo WBKEs (34) by ADM [34], OHAM [35] and SDM [36]. Figure (1) presents some curves of the numerical solutions $\tau(\nu_1, \nu_2)$ and $\pi(\nu_1, \nu_2)$ for (34) with $\nu_2 = 1$ and some values of ω. Similarly, Figures (2) and (3) present some curves of the numerical solutions $\tau(\nu_1, \nu_2)$ and $\pi(\nu_1, \nu_2)$ for (34) with $\nu_2 = 2$ and $\nu_2 = 3$, respectively. Figure (4)-(6) presents some surface plots of ARPST solutions for Caputo WBKEs (34) with $\omega = 1.0$, $\omega = 0.75$, $\omega = 0.5$, $\omega = 0.25$ and $\nu_2 = 1, 2, 3$.

6 Conclusions

We noticed from Tables (1), (2) that the approximate solutions of Caputo WBKEs (34) obtained from ARPST are not more accurate than the solutions resulting from SDM [36], but they are considered better and more accurate than the solutions resulting from the other methods ADM [34] and OHAM [35]. So anyway ARPST is considered efficient to examine the numerical solutions of Caputo WBKEs and its reliability provides approximate solutions that with numerical simulations present good discussion.

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Author Contributions

All authors contributed to the conception, analysis, writing, revision, and approval of the manuscript.

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Conflicts of Interest

The authors declare no conflict of interest.

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