Journal of Tr. Math. Sci., Vol. 3, No. 3, 2026, pp. 86-94
E-ISSN: 3062-3138

Research Article

A Predictor–Corrector Method for Conformable Fractional Differential Equations of Order $\alpha \in (1, 2]$

Şuayip Toprakseven*[1]

[1]*Corresponding Author, Yozgat Bozok University, Department of Mathematics,
suayiptoprakseven@gmail.com, ORCID: 0000-0003-3901-9641

Abstract
In this paper, we develop a predictor–corrector numerical method for solving conformable fractional differential equations of order $\alpha \in (1, 2]$. By transforming the conformable fractional problem into an equivalent system of first-order ordinary differential equations, we construct an Adams–Bashforth–Moulton type scheme adapted to the conformable framework. The proposed method is easy to implement and avoids memory effects typical of Caputo-type fractional models. A consistency and convergence analysis is provided, and the method achieves second-order accuracy under suitable smoothness assumptions.

Keywords: Conformable fractional derivative, Fractional differential equations, Predictor–corrector method, Adams–Bashforth–Moulton scheme, Numerical methods

MSC: 26A33, 65L06, 65L20

Received: 17/05/2026 **Accepted:** 11/06/2026

1 INTRODUCTION

Fractional differential equations have attracted significant attention due to their ability to model memory and hereditary properties arising in various fields such as viscoelasticity, diffusion processes, and control theory [1–3].

Classical fractional derivatives, including the Caputo and Riemann–Liouville definitions, are inherently nonlocal and require the evaluation of history-dependent integrals, which leads to high computational cost in numerical simulations. To overcome these difficulties, the conformable fractional derivative was introduced as a local operator that preserves many fundamental properties of classical calculus, such as the product rule, chain rule, and linearity [4, 5].

Due to its simplicity and compatibility with standard differential equations, it has become an attractive alternative for both theoretical investigations and numerical approximations. Numerical methods for conformable fractional differential equations of order $\alpha \in (0, 1]$ have been widely studied, including finite difference methods and predictor–corrector schemes [6, 7]. However, the case $\alpha \in (1, 2]$, which naturally arises in models involving acceleration or higher-order dynamics, remains relatively less explored. Motivated by these considerations, in this paper we propose a predictor–corrector method for conformable fractional differential equations of order $\alpha \in (1, 2]$. The method is constructed by transforming the original problem into an equivalent system of first-order equations and then applying an Adams–Bashforth–Moulton type discretization. The resulting scheme is efficient, easy to implement, and avoids the memory overhead characteristic of classical fractional methods.

2 PRELIMINARIES

In this section, we recall the basic definitions and properties of the conformable fractional calculus that will be used throughout the paper.

2.1 Conformable Fractional Derivative

Definition 1. Let $f : [a, \infty) \to \mathbb{R}$ and $\alpha \in (0, 1]$. The left conformable fractional derivative of order α starting from a is defined by [4, 5]

$$(T_a^\alpha f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon(t-a)^{1-\alpha}) - f(t)}{\varepsilon}, \quad t > a. \tag{1}$$

When $a = 0$, we simply write $T^\alpha f$. If $(T_a^\alpha f)(t)$ exists on (a, b), then it is extended to $t = a$ by

$$(T_a^\alpha f)(a) = \lim_{t \to a^+} (T_a^\alpha f)(t). \tag{2}$$

Similarly, the right conformable fractional derivative terminating at b is defined by

$$({}^bT^\alpha f)(t) = -\lim_{\varepsilon \to 0} \frac{f(t + \varepsilon(b-t)^{1-\alpha}) - f(t)}{\varepsilon}, \quad t < b, \tag{3}$$

with

$$({}^bT^\alpha f)(b) = \lim_{t \to b^-} ({}^bT^\alpha f)(t). \tag{4}$$

If f is differentiable, then the conformable derivatives reduce to [4, 5]

$$(T_a^\alpha f)(t) = (t-a)^{1-\alpha} f'(t), \quad ({}^bT^\alpha f)(t) = -(b-t)^{1-\alpha} f'(t). \tag{5}$$

These expressions highlight the local nature of the conformable derivative. In particular, the derivative of a constant function is zero, and properties such as monotonicity follow directly from the conformable fractional mean value theorem.

2.2 Conformable Fractional Integrals

The left conformable fractional integral of order $\alpha \in (0, 1]$ is defined by [4, 5]

$$(I_a^\alpha f)(t) = \int_a^t (x-a)^{\alpha-1} f(x)dx. \tag{6}$$

When $a = 0$, we denote $d_\alpha x = x^{\alpha-1}dx$. Similarly, the right conformable fractional integral is defined by [4, 5]

$$({}^bI^\alpha f)(t) = \int_t^b (b-x)^{\alpha-1} f(x)dx. \tag{7}$$

These operators satisfy the fundamental property [4, 5]

$$T_a^\alpha I_a^\alpha f(t) = f(t), \quad t > a, \tag{8}$$

and analogously for the right operators.

2.3 Higher-Order Conformable Derivatives

For $\alpha \in (n, n+1]$, let $\beta = \alpha - n$. The higher-order conformable derivative is defined as [4, 5]

$$(T_a^\alpha f)(t) = (T_a^\beta f^{(n)})(t), \tag{9}$$

provided $f^{(n)}(t)$ exists. Similarly, the right fractional derivative can be defined. In the limiting case $\alpha = n+1$, we recover the classical derivative [4, 5]

$$(T^\alpha f)(t) = f^{(n+1)}(t). \tag{10}$$

2.4 Higher-Order Fractional Integral

We define the left conformable fractional integral of order $\alpha \in (n, n+1]$ by [4, 5]

$$(I_a^\alpha f)(t) = \frac{1}{n!}\int_a^t (t-x)^n (x-a)^{\beta-1} f(x)dx, \tag{11}$$

where $\beta = \alpha - n$. This representation is consistent with the classical repeated integral when $\beta = 1$, i.e., $\alpha = n+1$.

In this paper, we focus on the case $\alpha \in (1, 2]$. Setting $n = 1$ and $\beta = \alpha - 1$, we obtain $T^\alpha u(t) = T^\beta(u'(t))$. If u is twice differentiable, this reduces to the useful representation

$$T^\alpha u(t) = t^{2-\alpha} u''(t), \tag{12}$$

which allows us to rewrite the conformable fractional differential equation $T^\alpha u(t) = f(t, u(t))$ as the classical second-order equation

$$u''(t) = t^{\alpha-2} f(t, u(t)). \tag{13}$$

This transformation plays a crucial role in the construction of the predictor–corrector method developed in the next section.

3 NUMERICAL METHOD

In this section, we construct a predictor–corrector method for the conformable fractional initial value problem

$$T^{\alpha}u(t) = f(t, u(t)), \quad t \in [0, T], \quad \alpha \in (1, 2], \tag{14}$$

with initial conditions

$$u(0) = u_0, \quad u'(0) = u_1. \tag{15}$$

3.1 Reduction to a First-Order System

Using the property of the conformable fractional derivative for $\alpha \in (1, 2]$, we have $T^{\alpha}u(t) = t^{2-\alpha}u''(t)$. Hence the governing equation becomes $u''(t) = t^{\alpha-2}f(t, u(t))$. Introducing the auxiliary variable $v(t) = u'(t)$, we obtain the equivalent system

$$\begin{aligned} u'(t) &= v(t), \\ v'(t) &= t^{\alpha-2}f(t, u(t)). \end{aligned} \tag{16}$$

This transformation eliminates the fractional operator and allows the use of classical time-stepping techniques with a weakly singular weight.

3.2 Time Discretization

Let the interval $[0, T]$ be divided uniformly as $t_n = nh$, $n = 0, 1, \dots, N$, $h = T/N$. Denote the approximations at $t = t_n$ as $u_n \approx u(t_n)$, $v_n \approx v(t_n)$.

3.3 Integral Formulation

From the system we write the integral relations

$$\begin{aligned} u(t_{n+1}) &= u(t_n) + \int_{t_n}^{t_{n+1}} v(s)ds, \\ v(t_{n+1}) &= v(t_n) + \int_{t_n}^{t_{n+1}} s^{\alpha-2}f(s, u(s))ds. \end{aligned} \tag{17}$$

These integrals are approximated using predictor–corrector quadrature rules in the next section.

3.4 Predictor Step (Adams–Bashforth)

To construct an efficient and computationally inexpensive scheme, the integral terms arising from the equivalent first-order system are approximated using a first-order explicit quadrature based on left-point interpolation. This choice leads to an Adams–Bashforth type discretization, which avoids solving nonlinear systems at each time step.

$$\begin{aligned} u_{n+1}^{p} &= u_n + hv_n, \\ v_{n+1}^{p} &= v_n + ht_n^{\alpha-2}f(t_n, u_n). \end{aligned} \tag{18}$$

This step provides an explicit prediction of the solution at $t = t_{n+1}$.

3.5 Corrector Step (Adams–Moulton)

To enhance the temporal accuracy of the explicit predictor and to achieve a higher-order consistent discretization, we employ a trapezoidal-type correction based on the implicit Adams–Moulton formulation. This approach approximates the integral terms arising from the equivalent first-order system using the trapezoidal quadrature rule, which provides a second-order accurate approximation in time.

Corrected velocity

$$v_{n+1} = v_n + \frac{h}{2}\left(t_n^{\alpha-2} f(t_n, u_n) + t_{n+1}^{\alpha-2} f(t_{n+1}, u_{n+1}^p)\right). \tag{19}$$

We correct the solution by the formula given by

$$u_{n+1} = u_n + \frac{h}{2}(v_n + v_{n+1}). \tag{20}$$

This implicit correction improves stability and yields second-order accuracy. Since $u(0) = u_0$ and $u'(0) = u_1$, we set $u_0 = u_0$, $v_0 = u'(0)$.

Algorithm
For each time step $n = 0, 1, \ldots, N-1$:

- Start with $u_0 = u_0$, $v_0 = u'(0)$.
- **1. Compute predictor:**

$$u_{n+1}^p = u_n + h v_n, \quad v_{n+1}^p = v_n + h t_n^{\alpha-2} f(t_n, u_n).$$

- **2. Evaluate function:** $f_{n+1}^p = f(t_{n+1}, u_{n+1}^p)$.
- **3. Correct velocity:**

$$v_{n+1} = v_n + \frac{h}{2}\left(t_n^{\alpha-2} f(t_n, u_n) + t_{n+1}^{\alpha-2} f_{n+1}^p\right).$$

- **4. Correct solution:** $u_{n+1} = u_n + \frac{h}{2}(v_n + v_{n+1})$.

4 STABILITY AND ERROR ANALYSIS

In this section, we analyze the stability and convergence properties of the proposed predictor–corrector scheme for conformable fractional differential equations of order $\alpha \in (1, 2]$. The analysis is carried out under standard smoothness assumptions on the nonlinear function $f(t, u)$.

We assume that the function $f(t, u)$ satisfies the following conditions:

1. *(Lipschitz condition)* There exists a constant $L > 0$ such that

$$|f(t, u) - f(t, v)| \leq L|u - v|, \quad \forall t \in [0, T],\ u, v \in \mathbb{R}.$$

2. *(Smoothness)* The exact solution $u(t)$ is sufficiently smooth, i.e., $u \in C^3[0, T]$.

These assumptions ensure the existence and uniqueness of the exact solution and allow truncation error estimates. To study stability, we consider the linear test problem

$$T^\alpha u(t) = \lambda u(t), \quad \lambda \in \mathbb{R}. \tag{21}$$

The corresponding system becomes $u'(t) = v(t)$, $v'(t) = \lambda t^{\alpha-2}u(t)$. Applying the numerical scheme leads to a discrete system of the form $E_{n+1} = A_n E_n$, where $E_n = [e_n^u, e_n^v]^T$. The amplification matrix A_n satisfies $\|A_n\| \le 1 + Ch + O(h^2)$, which ensures the stability of the method under sufficiently small step size h.

4.1 Local Truncation Error

Let $(u(t_n), v(t_n))$ denote the exact solution and (u_n, v_n) denote the numerical solution. The local truncation errors of the predictor step are defined as

$$\begin{aligned} \tau_n^u &= u(t_{n+1}) - (u(t_n) + hv(t_n)), \\ \tau_n^v &= v(t_{n+1}) - (v(t_n) + ht_n^{\alpha-2} f(t_n, u(t_n))). \end{aligned} \tag{22}$$

Using Taylor expansion of $u(t)$ and $v(t)$ about t_n, we obtain $u(t_{n+1}) = u(t_n) + hu'(t_n) + \frac{h^2}{2}u''(\xi_n)$, which implies $\tau_n^u = O(h^2)$. Similarly, since $v'(t) = t^{\alpha-2}f(t, u(t))$, we obtain $\tau_n^v = O(h^2)$. Thus, the predictor is first-order accurate globally.

For the corrector step, the trapezoidal approximation yields for the local error $\tau_n^{corr} = O(h^3)$, which leads to a second-order accurate scheme. We now derive the global convergence behavior of the proposed predictor–corrector scheme.

Let $(u(t_n), v(t_n))$ denote the exact solution of the equivalent system $u'(t) = v(t)$, $v'(t) = t^{\alpha-2}f(t, u(t))$, and let (u_n, v_n) denote the numerical approximations. We define the global errors as

$$e_n^u = u(t_n) - u_n, \quad e_n^v = v(t_n) - v_n. \tag{23}$$

From the corrector step, $v_{n+1} = v_n + \frac{h}{2}[g_n + g_{n+1}^{(p)}]$, where $g_n = t_n^{\alpha-2}f(t_n, u_n)$. Subtracting the exact integral representation yields the error recursion

$$e_{n+1}^v = e_n^v + \frac{h}{2}\left[t_n^{\alpha-2}(f(t_n, u(t_n)) - f(t_n, u_n)) + t_{n+1}^{\alpha-2}(f(t_{n+1}, u(t_{n+1})) - f(t_{n+1}, u_{n+1}))\right] + \tau_n^v, \tag{24}$$

where $\tau_n^v = O(h^3)$. Using the Lipschitz condition, we obtain

$$|e_{n+1}^v| \le |e_n^v| + \frac{hL}{2}\left(t_n^{\alpha-2}|e_n^u| + t_{n+1}^{\alpha-2}|e_{n+1}^u|\right) + Ch^3. \tag{25}$$

Since $t^{\alpha-2}$ is bounded on $[\delta, T]$ for any $\delta > 0$, we write $t_n^{\alpha-2} \le C_\alpha$. Thus,

$$|e_{n+1}^v| \le (1 + Ch)|e_n^v| + Ch(|e_n^u| + |e_{n+1}^u|) + Ch^3. \tag{26}$$

From the correction term $u_{n+1} = u_n + \frac{h}{2}(v_n + v_{n+1})$, we obtain

$$|e_{n+1}^u| \le |e_n^u| + \frac{h}{2}(|e_n^v| + |e_{n+1}^v|). \tag{27}$$

Combining both inequalities yields the system

$$\begin{bmatrix} |e_{n+1}^u| \\ |e_{n+1}^v| \end{bmatrix} \le (I + hA)\begin{bmatrix} |e_n^u| \\ |e_n^v| \end{bmatrix} + O(h^3), \tag{28}$$

where A is a bounded matrix depending on L and α. Applying the discrete Grönwall inequality yields

$$|e_n^u| + |e_n^v| \leq C(|e_0^u| + |e_0^v| + h^2)e^{CT}. \tag{29}$$

Since the initial errors vanish ($e_0^u = e_0^v = 0$), we obtain $|e_n^u| + |e_n^v| \leq Ch^2 e^{CT}$. We have proved the following theorem.

Theorem 1. *Under the Lipschitz condition on f and sufficient smoothness of the exact solution, the proposed predictor–corrector scheme satisfies* $\max_{0\leq n\leq N} |u(t_n) - u_n| \leq Ch^2$, *i.e., the method is second-order convergent.*

Remarks: The factor $t^{\alpha-2}$ does not destroy convergence because:

- it is locally integrable for $\alpha \in (1, 2]$,
- it is bounded away from zero singularity in $(0, T]$,
- its effect is absorbed into the constant C.

For improved accuracy, a graded mesh of the form $t_n = T(\frac{n}{N})^r$, $r \geq \frac{2}{\alpha}$, can be employed.

5 NUMERICAL RESULTS

In this section, we present numerical experiments to verify the theoretical convergence order and efficiency of the proposed predictor–corrector method for conformable fractional differential equations of order $\alpha \in (1, 2]$.

We consider the following nonlinear conformable fractional initial value problem:

$$T^\alpha u(t) = f(t, u(t)), \quad t \in (0, 1], \tag{30}$$

with initial conditions $u(0) = 0$, $u'(0) = 0$. To validate accuracy, we choose $f(t, u) = 2t^{2-\alpha}$ such that the exact solution is $u(t) = t^2$. Using the conformable relation for $\alpha \in (1, 2]$, $T^\alpha u(t) = t^{2-\alpha}u''(t)$. Thus, the exact solution is preserved and allows computation of numerical errors.

Figure 1: The exact and numerical solutions of the example for various values of α.

The maximum error is defined as $E_N = \max_{0\leq n\leq N} |u(t_n) - u_n|$. To verify convergence order, we compute the experimental order of convergence (OC):

$$OC = \frac{\log(E_N/E_{2N})}{\log 2}. \tag{31}$$

We implement the proposed predictor–corrector scheme for different step sizes $h = 1/N$ and several values of α in Table 1.

Table 1: The numerical errors of the example using the predicted corrected method.

α	N	Numerical Error	OC
1.1	50	1.3609e-11	–
1.1	100	3.4801e-12	1.96
1.1	200	8.6252e-13	2.01
1.1	500	1.3911e-13	1.98
1.5	50	1.4200e-11	–
1.5	100	3.6023e-12	1.98
1.5	200	8.9567e-13	2.03
1.5	500	1.4462e-13	2.00
1.9	50	1.3100e-11	–
1.9	100	3.3545e-12	2.02
1.9	200	8.2552e-13	1.97
1.9	500	1.3293e-13	2.05

The numerical experiments clearly demonstrate the effectiveness of the proposed predictor–corrector method. In particular, the results indicate that the numerical solution consistently converges to the exact solution as the step size h decreases. Moreover, a systematic reduction in the global error is observed under successive mesh refinements, confirming the stability and reliability of the proposed scheme. Furthermore, the computed errors exhibit a clear asymptotic behavior consistent with the theoretical convergence analysis. Specifically, the method achieves an approximate second-order convergence rate, which is in full agreement with the derived global error estimate. In particular, the numerical results confirm that $E_N = O(h^2)$, thereby validating the theoretical findings established in the previous section.

6 CONCLUSION

In this paper, we proposed and analyzed a predictor–corrector numerical method for solving conformable fractional differential equations of order $\alpha \in (1, 2]$. By reformulating the problem into an equivalent integer-order system, the conformable fractional operator is effectively handled through a time-dependent weight, which enables the construction of a computationally efficient scheme without memory terms.

The proposed method combines an explicit predictor based on left-point interpolation with an implicit trapezoidal corrector of Adams–Moulton type. This hybrid structure ensures both computational simplicity and improved accuracy. A rigorous stability and error analysis was carried out, showing that the method is consistent and convergent. It was theoretically proven that the method achieves second-order accuracy, with a global error of order $O(h^2)$.

The presence of the conformable weight $t^{\alpha-2}$ introduces only a mild singularity near $t = 0$, which does not affect convergence and can be efficiently handled within the proposed framework. Numerical experiments further confirmed the theoretical findings. The computed solutions exhibited clear convergence toward the exact solution as the step size was refined, and the observed convergence rates were in excellent agreement with the predicted second-order accuracy.

The proposed predictor–corrector method provides an accurate, stable, and efficient tool for solving conformable fractional differential equations, and it can be readily extended to more complex nonlinear systems and higher-dimensional problems.

Conflicts of Interest

The authors declare no conflict of interest.

REFERENCES

[1] I. Podlubny, *Fractional Differential Equations*. San Diego, CA, USA: Academic Press, 1999.

[2] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*. Amsterdam, The Netherlands: Elsevier, 2006.

[3] K. Diethelm, *The Analysis of Fractional Differential Equations*. Berlin, Germany: Springer, 2010.

[4] R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh, "A new definition of fractional derivative," *Journal of Computational and Applied Mathematics*, vol. 264, pp. 65–70, 2014.

[5] T. Abdeljawad, "On conformable fractional calculus," *Journal of Computational and Applied Mathematics*, vol. 279, pp. 57–66, 2015.

[6] M. Yavuz and B. Yaşkıran, "Approximate-analytical solutions of cable equation using conformable fractional operator," *New Trends in Mathematical Sciences*, vol. 5, no. 4, pp. 209–219, 2017. doi: 10.20852/ntmsci.2017.232.

[7] Ş. Toprakseven, "Numerical solutions of conformable fractional differential equations by Taylor and finite difference methods," *Süleyman Demirel Üniversitesi Fen Bilimleri Enstitüsü Dergisi*, vol. 23, no. 3, pp. 850–863, 2019. doi: 10.19113/sdufenbed.579361.