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On Some Weighted Inequalities on Time Scale with Nabla Calculus

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Abstract

The theory of time scales is one of the important cornerstones of functional analysis and operator theory. Recently, it has been the subject of many studies from different disciplines. For example, it has become the field of study of many researchers working in mathematics, economics, physics, optics, engineering, and other fields. In this study, firstly the basic features of the time scale and nabla calculus are mentioned. Then a new approach to the weighted Ostrowski-type inequality is presented using nabla calculus on time scales.

Keywords: ***Weighted function, Time scale, Ostrowski inequality.***

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1. INTRODUCTION

Inequality theory has an important place in almost every field of mathematics. Clear results cannot always be achieved in mathematical problems. In these cases, it is necessary to develop approach methods.The basic Ostrowski inequality was expressed by Ostrowski in 1938 [6] as follows:

Let $f: [\mathrm{a}, \mathrm{b}] \subseteq \mathrm{R} \rightarrow \mathrm{R}$ be a differentiable function on (a, b) and continuos on [a,b] and

f' : $(a, b) \to R$ is bounded on (a,b). Then for each $x \in [a, b]$

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \le \left[\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty \quad (1.1)$$

inequality holds.

Here $\|f'\|_\infty = \sup_{x\in(a,b)}|f'(x)|$ and $\frac{1}{4}$ is the best constant (Ostrowski, 1938).

The Ostrowski inequality sets an upper bound for the approximation of the integral mean $\int_a^b f(t)dt$ with the help of the value of f(x) in the closed interval $x \in [a, b]$.

[37]Bohner and Matthews proved the Ostrowski inequality using Montgomery's inequality on the time scale. In addition, continuous case, discrete case, quantum analysis case and weighted Ostrowski versions of the results are also given.

The Ostrowski-type inequality is found to be an exalted and applicable tool in several branches of mathematics. Integral inequalities find comprehensive applications in operator theory, statistics, probability theory, numerical integration, nonlinear analysis, information theory, stochastic analysis, approximation theory, biological sciences, physics and technology.So there is a lot of research, some precedents are as follows,
[38-39]Ostrowski type inequalities for functions whose first derivatives belong to the spaces$L_p[\alpha, \beta]$ and $L_1[\alpha, \beta]$ were presented by Dragomir and Wang.

[40]An Ostrowski-type inequality for functions with limited second derivatives was found by Cerone etc..

[41]Dragomir gave important Ostrowski-type inequalities for absolutely continuous functions.

[42]Sarıkaya obtained Ostrowski-type inequalities containing Riemann-Liouville fractional integrals for functions with limited second derivatives.

[43] In his study of the Ostrowski inequality, Anderson also proved versions of many fundamental inequalities such as Hermite-Hadamard, Steffensen and Chebyshev, which include compatible fractional integrals.

Our main purpose in this study is to bring a different perspective to the weighted Ostrowski inequality.

2. MATERIALS AND METHODS

Stefan HILGER laid the foundations of the study of dynamic equations on time scales in 1988. The main purpose of studying time scale is based on the idea of combining discrete analysis and continuous analysis. For more information on time scales and inequalities, we refer the reader to monographs [1-45].

Definition 2.1 [1] Any closed subset of the set of real numbers is called a time scale and is denoted $\mathbb{T}$. The metric on this set will be taken as the usual metric at $\mathbb{R}$.

Example 2.1 Sets such as $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$, $[2, 4] \cup \{6\}$, $\{a\}$ are time scales. However $\mathbb{Q}$, $\mathbb{C}$, $(0, 2)$, $(a, b]$ are not time scales.

Definition 2.2 [1] Let $\mathbb{T}$ be a time scale and $t \in \mathbb{T}$. Forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$

$$\sigma(t) = \begin{cases} inf\{s \in \mathbb{T} : s > t\}, & t \neq sup\mathbb{T} \\ t, & t = sup\mathbb{T} \end{cases} \tag{2.1}$$

Backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$

$$\rho(t) = \begin{cases} sup\{s \in \mathrm{T}: s < t\}, & t \neq inf\mathbb{T} \\ t, & t = inf\mathbb{T} \end{cases} \tag{2.2}$$

Definition 2.3 [1] Let $\mathbb{T}$ be a time scale. In this situation

$$\mu: \mathbb{T} \to [0, \infty), \mu(t) = \sigma(t) - t \tag{2.3}$$
$$\eta: \mathbb{T} \to [0, \infty), \eta(t) = t - \rho(t) \tag{2.4}$$

The functions μ and η are called graininess functions.

Definition 2.4 [1] A point $t \in \mathbb{T}$

(i) If $t < sup\mathbb{T}$ and $\sigma(t) > t$ then t is the right-scattered point,

(ii) If $t > \inf\mathbb{T}$ and $\rho(t) < t$ then t is left-scattered point,

(iii) If t is left-scattered and right-scattered point then t is isolated point,

(iv) If $t < \sup\mathbb{T}$ and $\sigma(t) = t$ then t is right-dense point,

(v) If $t > \inf\mathbb{T}$ and $\rho(t) = t$ then t is left-dense point,

(vi) If t is right-dense and left-dense point then t is dense point.

t right-scattered	$t < \sigma(t)$
t right-dense	$t = \sigma(t)$
t left-scattered	$\rho(t) < t$
t left-dense	$\rho(t) = t$
t isolated	$\rho(t) < t < \sigma(t)$
t dense	$\rho(t) = t = \sigma(t)$

Definition 2.5 [4] Let $\mathbb{T}$ be a time scale, $f : \mathbb{T} \to \mathbb{R}$ a function and $t \in \mathbb{T}_K$. If $L \in \mathbb{R}$ for all $\varepsilon > 0$

$$|f(\rho(t)) - f(s) - L[\rho(t) - s]| \leq \varepsilon|\rho(t) - s|$$

If a neighborhood $U_\delta(t) = (t-\delta, t+\delta) \cap \mathbb{T}$ can be found to satisfy the inequality for all $s \in U_\delta(t)$ then the number $L \in \mathbb{R}$ is called the nabla derivative of the f function at the point $t \in \mathbb{T}_k$ and this situation it is donated by $L = f^\nabla(t)$

Theorem 2.1 [3] Let $f: \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_k$. Thus

(i) If the function f is differentiable at point t, then the function f is continuous at point t.

(ii) If the function f is continious at the point t and left-scattered at the point t, then the function f is nabla differantiable at the point t

(iii) $$f^\nabla(t) = \frac{f(t)-f(\rho(t))}{\eta(t)} \tag{2.5}$$

(iv) If the point t is left dense, the function is nabla differantiable at the point and
$$f^\nabla(t) = \lim_{s\to t} \frac{f(t)-f(s)}{t-s} \tag{2.6}$$

(v) If the function f is differantiable at the point t
$$f^\rho(t) = f(t) - \eta(t) f^\nabla(t) \tag{2.7}$$

Theorem 2.2 [3] Let $f, g: \mathbb{T} \to \mathbb{R}$ be defined as follows for differantiable at point $t \in \mathbb{T}_k$. In this situation

(i) The functions $f+g: \mathbb{T} \to \mathbb{R}$ is also nabla differantiable at point t and
$$(f+g)^\nabla(t) = f^\nabla(t) + g^\nabla(t) \tag{2.8}$$

(ii) For any constant α, the function $\alpha f: \mathbb{T} \to \mathbb{R}$ is also nabla differantiable and
$$(\alpha f)^\nabla(t) = \alpha f^\nabla(t). \tag{2.9}$$

(iii) The function $f, g: \mathbb{T} \to \mathbb{R}$ is also nabla differantiable and
$$(fg)^\nabla(t) = f(t) g^\nabla(t) + f^\nabla(t) g(\rho(t)) = f^\nabla(t) g(t) + f(\rho(t)) g^\nabla(t). \tag{2.10}$$

(iv) If $f(t)f(\rho(t)) \neq 0$, the $\frac{1}{f}$ function is nabla differantiable and

$$\left(\frac{1}{f}\right)\nabla(t) = -\frac{f\nabla(t)}{f(t)f(\rho(t))} \tag{2.11}$$

(v) If $g(t)g(\rho(t)) \neq 0$ $\frac{f}{g}$ function is nabla differantiable and
$$\left(\frac{f}{g}\right)^\nabla(t) = \frac{f^\nabla(t)g(t)-f(t)g^\nabla(t)}{g(t)g(\rho(t))} \tag{2.12}$$

Theorem 2.3 [3] If the function $f: \mathbb{T} \to \mathbb{R}$ is continuous in $[a,b]$ and differantiable $[a,b)$

$$f^\nabla(\xi) \leq \frac{f(b)-f(a)}{b-a} \leq f^\nabla(\xi') \tag{2.13}$$

where $\xi, \xi' \in [a,b)$.

Definition 2.6 [3] If the function $f\ :\mathbb{T} \to \mathbb{R}$ is regular, there is a function F such that $F^{\nabla}(t) = f(t)$ and the function F is called the antiderivative of the function f .

Definition 2.7 [3] If the condition $F^{\nabla}(t) = f(t)$ is satisfied for $f\ :\mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_k$, the function $F:\mathbb{T} \to \mathbb{R}$ is called the nabla antiderivative of the function f and the nabla integral of the functions where c is an optional constant.

$$\int f(t)\nabla t = F(t) + c \tag{2.14}$$

Nabla integral of f from a to b for all $a, b \in \mathbb{T}$. It is of the form,

$$\int_a^b f(t)\nabla t = F(b) - F(a) \tag{2.15}$$

Definition 2.8 [3] Let $f\ :\mathbb{T} \to \mathbb{R}$ be a function. If the function f has continious limits at the dense points on $\mathbb{T}$ and right limits at the right dense points on $\mathbb{T}$, the function f is called and ld-continious function.

Theorem 2.4 [3] If $f\ :\mathbb{T} \to \mathbb{R}$ ld-continious and $t \in \mathbb{T}_k$, in this situation

$$\int_{p(t)}^{t} f(t)\nabla t = f(t)\eta(t) \tag{2.16}$$

Theorem 2.5 [3] If $a\,, b\,, c \in \mathbb{T}\,, \alpha \in \mathbb{R}$ and $f\,, g : \mathbb{T} \to \mathbb{R}$ ld-continuous in this situation, then the following statements are holds.

(i) $\int_a^b [f(t) + g(t)]\nabla t = \int_a^b f(t)\nabla t + \int_a^b g(t)\nabla t$ (2.17)

(ii) $\int_a^b (af)(t)\nabla t = a\int_a^b f(t)\nabla t$ (2.18)

(iii) $\int_a^b f(t)\nabla t = -\int_b^a f(t)\nabla t$ (2.19)

(iv) $\int_a^b f(t)\nabla t = \int_a^c f(t)\nabla t + \int_c^b f(t)\nabla t$ (2.20)

(v) $\int_a^b f(p(t))\, g^{\nabla}(t)\nabla t = (f\,g)(b) - (fg)(a) - \int_a^b f^{\nabla}(t)\, g(t)\nabla t$ (2.21)

(vi) $\int_a^b f(t)\, g^{\nabla}(t)\nabla t = (f\,g)(b) - (fg)(a) - \int_a^b f^{\nabla}(t)\, g(p(t))\nabla t$ (2.22)

(vii) $\int_a^a f(t)\nabla t = 0$ (2.23)

Let $t \in \mathbb{T}_K^K$ and $f\ :\mathbb{T} \to \mathbb{R}$. In this instance the existence of the delta derivative of the function f at t does not mean that the nabla derivative also exists. The opposite is also true.

Theorem 2.6 [5] (Hölder's inequality). Let $\kappa, \ell \in \mathrm{T}$ and $g, h: [\kappa, \ell] \longrightarrow \mathbb{R}$ be ld-continuous.Then

$$\int_{\kappa}^{\ell} |g(\theta)h(\theta)|\nabla\theta \le \left(\int_{\kappa}^{\ell} |g(\theta)|^p \nabla\theta\right)^{\frac{1}{p}} \left(\int_k^{\ell} |h(\theta)|^q\, \nabla\theta\right)^{\frac{1}{q}} \tag{2.24}$$

where $1 < p$ and $\frac{1}{p} + \frac{1}{q} = 1$.

To prove Theorem 3.1, we require the below generalized Montgomery identity.

Lemma 2.1 (the generalized Montgomery identity). Let $\kappa,\ell,\tau,\theta \in \mathrm{T}$, $\kappa < \ell$ and $g:[\kappa,\ell] \longrightarrow \mathbb{R}$ be differantiable and parameter $\lambda \in [0,1]$. Then

$$(1-\lambda)g(\theta)+\frac{\lambda}{2}\big(g(\kappa)+g(\ell)\big)=\frac{1}{\ell-\kappa}\int_{\kappa}^{\ell} g^{\rho}(\tau)\nabla\tau+\frac{1}{\ell-\kappa}\int_{\kappa}^{\ell} p(\theta,\tau)g^{\nabla}(\tau)\nabla\tau, \qquad (2.25)$$

where

$$p(\theta,\tau)=\begin{cases}\tau-\left(\kappa+\lambda\dfrac{\ell-\kappa}{2}\right), \kappa\le\tau<\theta,\\ \tau-\left(\ell-\lambda\dfrac{\ell-\kappa}{2}\right), \theta\le\tau\le\ell.\end{cases}$$

Lemma (2.1) can be done by applying Theorem (2.6).

3. RESULTS

Theorem 3.1 (The generalized Ostrowski inequality). Let $\kappa,\ell,\tau,\theta \in \mathrm{T}$, $\kappa < \ell$ and $g:[\kappa,\ell] \longrightarrow \mathbb{R}$ be differentiable and parametre $\lambda \in [0,1]$, then we have

$$\left|(1-\lambda)g(\theta)+\frac{\lambda}{2}\big(g(\kappa)+g(\ell)\big)-\frac{1}{\ell-\kappa}\int_{\kappa}^{\ell} g^{\rho}(\tau)\nabla\tau\right|\le\frac{\mathrm{M}}{\ell-\kappa}\big(f_2(\theta,\kappa)+f_2(\theta,\ell)\big), \quad (3.1)$$

where $M=\inf_{\kappa<\theta<\ell}|g^{\nabla}(\theta)|$.

This inequality is sharp in the sense that the right-hand side of (3.1) can not be replaced by a smaller one.

Using Lemma 2.1(the generalized Montgomery identity) with $p(\theta,\tau)$, we have

$$\begin{aligned}\left|(1-\lambda)g(\theta)+\frac{\lambda}{2}\big(g(\kappa)+g(\ell)\big)-\frac{1}{\ell-\kappa}\int_{\kappa}^{\ell} g^{\rho}(\tau)\nabla\tau\right| &= \left|\frac{1}{\ell-\kappa}\int_{\kappa}^{\ell} p(\theta,\tau)g^{\nabla}(\tau)\nabla\tau\right| \qquad (3.2)\\ &\le\frac{\mathrm{M}}{\ell-\kappa}\left(\int_{\kappa}^{\theta}\left|\tau-\left(\kappa+\lambda\frac{\ell-\kappa}{2}\right)\right|\nabla\tau+\int_{\theta}^{\ell}\left|\tau-\left(\ell-\lambda\frac{\ell-\kappa}{2}\right)\right|\nabla\tau\right)\\ &=\frac{\mathrm{M}}{\ell-\kappa}\left[\int_{\kappa}^{\theta}\left(\tau-\left(\kappa+\lambda\frac{\ell-\kappa}{2}\right)\right)\nabla\tau+\int_{\theta}^{\ell}\left(\left(\ell-\lambda\frac{\ell-\kappa}{2}\right)-\tau\right)\nabla\tau\right]\\ &=\frac{\mathrm{M}}{\ell-\kappa}\big(f_2(\theta,\kappa)+f_2(\theta,\ell)\big).\end{aligned}$$

The below weighted Ostrowski inequality with parameter on time scale holds.

Theorem 3.2 Presume the assumptions of Theorem 3.1 is true and $\xi \in T$ and $q \in C_{\ell d}$.

Then

$$\left| A + (1-\lambda)g(\theta) - \int_{\kappa}^{\ell} q^{\rho}(\tau) g^{\rho}(\tau) \nabla\tau \right|$$

$$\leq A + \int_{\kappa}^{\vartheta} q^{\rho}(\tau)(1-\lambda)g(\theta) - g^{\rho}(\tau)\nabla\tau + \int_{\vartheta}^{\ell} q^{\rho}(\tau)(1-\lambda)g(\theta) - g^{\rho}(\tau)\nabla\tau \tag{3.3}$$

$$\leq \begin{cases} A + (1-\lambda)\left(\int_{\kappa}^{\ell} |g(\theta)|^{p}\nabla\tau\right)^{\frac{1}{p}} \left(\int_{\kappa}^{\ell} \left(q^{\rho}(\tau)\right)^{q}\nabla\tau\right)^{\frac{1}{q}} + \left(\int_{\kappa}^{\ell} |g^{\rho}(\tau)|^{p}\nabla\tau\right)^{\frac{1}{p}} \left(\int_{\kappa}^{\ell} \left(q^{\rho}(\tau)\right)^{q}\nabla\tau\right)^{\frac{1}{q}} \text{ where } \frac{1}{p}+\frac{1}{q}=1, p>1, \\ A + \underset{\kappa \leq \tau < \ell}{inf}\ q^{\rho}(\tau)[h_2((\kappa,\theta) + h_2(\ell,\theta)] \\ \frac{g(\rho(\ell) - g(\rho(\kappa))}{2} + \left|(1-\lambda)g(\theta) - \frac{g(\rho(\kappa) + g(\rho(\ell))}{2}\right|, \end{cases} \tag{3.4}$$

where $A = \frac{\lambda}{2}\big(g(\kappa) + g(\ell)\big)$, $\int_{\kappa}^{\ell} q^{\rho}(\tau)\nabla\tau = 1$, $q(\tau) \geq 0$.

As from left side of (3.1) we have

$$\left| \frac{\lambda}{2}(g(\kappa) + g(\ell)) + (1-\lambda)g(\theta) - \int_{\kappa}^{\ell} q^{\rho}(\tau) g^{\rho}(\tau)\nabla\tau \right|$$

$$= \left| \frac{\lambda}{2}(g(\kappa) + g(\ell)) + \int_{\kappa}^{\vartheta} q^{\rho}(\tau)\big((1-\lambda)g(\theta) - g^{\rho}(\tau)\big)\nabla\tau \right|$$

$$\leq A + \int_{\kappa}^{\vartheta} q^{\rho}(\tau)|(1-\lambda)g(\theta) - g^{\rho}(\tau)|\nabla\tau + \int_{\vartheta}^{\ell} q^{\rho}(\tau)|(1-\lambda)g(\theta) - g^{\rho}(\tau)|\nabla\tau$$

and therefore (3.3) is shown. The first part of (3.4) can be done easily by applying Hölder's inequality. By factoring $\underset{\kappa\leq\tau<\ell}{inf}\ q^{\rho}(\tau)$, we have

$$A + \int_{\kappa}^{\vartheta} q^{\rho}(\tau)|(1-\lambda)g(\theta) - g^{\rho}(\tau)|\nabla\tau + \int_{\vartheta}^{\ell} q^{\rho}(\tau)|(1-\lambda)g(\theta) - g^{\rho}(\tau)|\nabla\tau$$

$$\leq A + \underset{\kappa\leq\tau<\ell}{inf}\ q^{\rho}(\tau)\left(\int_{\kappa}^{\vartheta}(q^{\rho}(\tau) - (1-\lambda)g(\theta))\nabla\tau + \int_{\vartheta}^{\ell}((1-\lambda)g(\theta) - g^{\rho}(\tau))\nabla\tau\right)$$

$$= A + \underset{\kappa\leq\tau<\ell}{inf}\ q^{\rho}(\tau)\left(\int_{\vartheta}^{\kappa}\big((1-\lambda)g(\theta) - g^{\rho}(\tau)\big)\nabla\tau + \int_{\vartheta}^{\ell}\big((1-\lambda)g(\theta) - g^{\rho}(\tau)\big)\nabla\tau\right)$$

$$= A + \underset{\kappa\leq\tau<\ell}{inf}\ q^{\rho}(\tau)[h_2(\kappa,\theta) + h_2(\ell,\theta)]$$

and in this way the 2nd part of (3.4) holds. Eventually for deriving the 3rd inequality, we apply the fact that

$$\inf_{\kappa \leq \tau < \ell} \{|g(\rho(\tau)) - (1-\lambda)g(\theta)|\}$$

$$= \min\{g(\rho(\ell)) - (1-\lambda)g(\theta), (1-\lambda)g(\theta) - g(\rho(k))\}$$

$$= \frac{g(\rho(\ell)) - g(\rho(k))}{2} + \left|(1-\lambda)g(\theta) - \frac{g(\rho(\kappa) + g(\rho(\ell))}{2}\right|.$$

Thus (3.4) is shown.

Remark 3.1. If we put $q^{\rho(\tau)} = \frac{1}{\ell-\kappa}$ in Theorem 3.2 then obtain the result without weights.

Remark 3.2. If we put $q^{\rho(\tau)} = \frac{1}{\ell-\kappa}$ and $\lambda = 0$ in Theorem 3.2 then we recapture the Theorem 3.1 of [42]

4. DISCUSSION

There are some studies on the Ostrowski inequality on the time scales. In this study, the generalized weighted Ostrowski inequalities are proved on time scales with nabla calculus. As a result, germane readers will be able to discover new inequalities and application areas through the results obtained in this study. Also in further studies, these inequalities can be studied for functions of more variables or in other parts of time scales.

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