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A new approach to Otrowski inequalities on time scale with Nabla calculus

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Abstract

Recently, the theory of time scales has come to the fore in many areas of the scientific world. It has taken place in the field of many researchers working in mathematics, physics, engineering, economics, optics, and other fields. This study brings a new approach to the generalized Ostrowski inequalities with the nabla calculus.

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1. INTRODUCTION

Time scales, introduced by Stefan Hilger, were defined as a field and contributed to the development of classical analysis. Time scales provided a common expression for analyzing discrete and continuous situations. If real numbers are used in time scales, continuous analysis can be mentioned, and if integers are used, discrete analysis can be mentioned. Time scales have been used in many disciplines recently. Examples of these are mathematics, economics, and physics. In this study, the use of the Nabla derivative in time scales will be discussed. Information on the Nabla derivative is given in the second section.

The Montgomery Identity is expressed below.

$c, d, s, r \in \mathbb{T}, c < l$ and $f: [c, d] \to \mathbb{R}$ let be a defined function.

$p(r,s) = \begin{cases} s-c & c \le s < r \\ s-d & r \le s \le d \end{cases}$ to be

$$f(r) = \frac{1}{d-c}\int_c^d f^{\sigma}(s)\Delta s + \frac{1}{d-c}\int_c^d p(r,s) f^{\Delta}(s)\Delta s.$$

Ostrowski proved the following integral inequality in 1938 as follows.

$c, d, s, r \in \mathbb{T}, c < d$ and $f: [c, d] \to \mathbb{R}$ let be a defined and differentiable function.

$$\left| f(r) - \frac{1}{d-c}\int_c^d f^{\sigma}(s) s \right| \le \frac{K}{d-c}\left(g_2(r,c) + g_2(r,d)\right)$$

inequality is reached. Here

$$K = \sup_{c < r < d} \left| f^{\Delta}(r) \right|.$$

There have been many studies on Ostrowski's inequality in the last 10 years. Some of these studies have been done on convex functions, fractional integrals, and double integrals.

Our main purpose in this study is to bring a different perspective to Ostrowski inequality with the nabla calculation.

1. Preliminaries

Now let us give the following concepts to prove our basic results. For more information on time scales and inequalities, we refer the reader to monographs [1-22].

Definition 2.1[8] Any closed subset of the set of real numbers is called a time scale. This cluster is denoted by $\mathbb{T}$. $\mathbb{R}, \mathbb{Z}, \mathbb{N}, [1,3]$ sets can be given as examples of time scales. However, $\mathbb{C}$, $(0,1)$ sets are not time scales.

Definition 2.2[8] Let the set $\mathbb{T}$ be given as a time scale and select an element such that $a \in \mathbb{T}$.

The forward jump operator is defined as follows:

$$\sigma: \mathbb{T} \longrightarrow \mathbb{T}$$

$$\sigma(a) = \begin{cases} inf\{s \in \mathbb{T}: s > a\}, & a \neq sup\mathbb{T} \\ a, & a = sup\mathbb{T} \end{cases}$$

$c, d, s, r \in \mathbb{T}, c < d$ ve $f: [c, d] \to \mathbb{R}$ be the defined function.

$p(r,s) = \begin{cases} s - c & c \leq s < r \\ s - d & r \leq s \leq d \end{cases}$ let's take it as.

$$f(r) = \frac{1}{d-c}\int_{c}^{d} f^{\rho}(s)\nabla s + \frac{1}{d-c}\int_{c}^{d} p(r,s) f^{\nabla}(s)\nabla s.$$

Backward jump operator is defined as follows:

$$\rho: \mathbb{T} \longrightarrow \mathbb{T}$$

$$\rho(a) = \begin{cases} sup\{s \in \mathbb{T}: s < a\}, & a \neq inf\mathbb{T} \\ a, & a = inf\mathbb{T} \end{cases}$$

Definition 2.3 [3] Let's take a point $a \in \mathbb{T}$.

i) If $a < sup\mathbb{T}$ and $\sigma(a) > a$ then a right scattering point,
ii) If $a > inf\mathbb{T}$ ve $\rho(a) < a$ then a left scattering point,
iii) If it is a point with right and left scattering, it is an isolated point,
iv) If $a < sup\mathbb{T}$ and $\sigma(a) = a$ then a right dense point,
v) If $a > inf\mathbb{T}$ and $\rho(a) = a$ then a left dense point,
vi) If the right and left dense points are called dense points.

Definition 2.4 [8] The Nabla differentiability region is $\mathbb{T}_{\kappa} = \mathbb{T} - \{s\}$,if $\mathbb{T}$ has a right-scattered minimum s, otherwise $\mathbb{T}_{\kappa} = \mathbb{T}$.

Definition 2.5 [3] The function f is defined in $\mathbb{T}$ as follows:

$$f^{\rho}(a) = f(p(a))$$

Definition 2.6 [3]For every $a \in \mathbb{T}$;

$$v: \mathbb{T} \longrightarrow [0, \infty]$$

$$\upsilon(a) = a - \rho(a)$$

the function is called as backward graininess function.

Theorem 2.7 [3]Let $f: \mathbb{T} \longrightarrow \mathbb{R}$ be a function and $a \in \mathbb{T}_\kappa$. In this case, we can write the following expressions;

i) If f is differentiable to nabla at a point a, then f is continuous at point a.

ii) If f is continuous at point a and point a is scattered to the left, then f is differentiable to nabla at point a.

$$f^\nabla(a) = \frac{f(a) - f(b)}{\upsilon(a)}.$$

iii) If a is the left-dense point, then f is nabla differentiable at a.

$$\lim_{b \longrightarrow a} \frac{f(a) - f(b)}{a - b}$$

the expression exists as a finite number. Then

$$f^\nabla(a) = \lim_{b \longrightarrow a} \frac{f(a) - f(b)}{a - b}.$$

iv) If f is differentiable to nabla at point a, then

$$f^\rho(a) = f(a) - \upsilon(a) f^\nabla(a).$$

Theorem 2.8[3] Let the functions $f, g: \mathbb{T} \longrightarrow \mathbb{R}$, f, g be differentiable to nabla and $a \in \mathbb{T}_\kappa$.

i) The expression $f + g: \mathbb{T} \longrightarrow \mathbb{R}$ is nabla differentiable at point a.

$$(f + g)^\nabla(a) = f^\nabla(a) + g^\nabla(a).$$

ii) For any constant α, the expression $\alpha f: \mathbb{T} \longrightarrow \mathbb{R}$ is nabla differentiable.

$$(\alpha f)^\nabla(a) = \alpha f^\nabla(a).$$

iii) The expression $fg: \mathbb{T} \longrightarrow \mathbb{R}$ is nabla differentiable and the following rule can be used.

$$(fg)^\nabla(a) = f^\nabla(a) g(a) + f^\rho(a) g^\nabla(a) = f(a) g^\nabla(a) + f^\nabla(a) g^\rho(a).$$

iv) If $f(a) f^\rho(a) \neq 0$ then the expression $\frac{1}{f}$ is nabla differentiable at a.

$$\left(\frac{1}{f}\right)^{\nabla}(a) = -\frac{f^{\nabla}(a)}{f(a)f^{\rho}(a)}.$$

v) If $g(t)g^{\rho}(t) \neq 0$, the expression $\frac{f}{g}$ is Nabla differentiable at a and the following quotient rule is obtained

$$\left(\frac{f}{g}\right)^{\nabla}(a) = \frac{f^{\nabla}(a)g(a) - f(a)g^{\nabla}(a)}{g(a)g^{\rho}(a)}.$$

Definition 2.9[3] Provided that the function given as $F: \mathbb{T} \longrightarrow \mathbb{R}$ satisfies the expression $F^{\nabla}(a) = f(a)$ at all $a \in \mathbb{T}_{\kappa}$ points, the function F is called the nabla anti-derivative of the function f. Then we define the integral of f as follows;

$$\int_{b}^{a} f(a)\nabla a = F(a) - F(b).$$

Definition 2.10 [8] Let the function $f: \mathbb{T} \longrightarrow \mathbb{R}$ be given. The function f is ld-continuous or left-dense continuous if it is continuous at all left-dense points in $\mathbb{T}$ and $\lim_{b \longrightarrow a^{+}} f(b)$ exists as a finite number for all right-dense points $a \in \mathbb{T}$ It is called.

Theorem 2.11 [17] The ld-continuous functions have a nabla anti-derivative.

Theorem 2.12 [3] If $f: \mathbb{T} \longrightarrow \mathbb{R}$ function is ld-continuous and $a \in \mathbb{T}_{\kappa}$.

$$\int_{\rho(a)}^{a} f(a)\nabla a = f(a)\upsilon(a).$$

Theorem 2.13 [3] Let $x, y, z \in \mathbb{T}$, $\alpha \in \mathbb{R}$ and $f, g: \mathbb{T} \longrightarrow \mathbb{R}$ function be ld-continuous.

i) $\int_{x}^{y} [f(a) + g(a)]\nabla a = \int_{x}^{y} f(a)\nabla a + \int_{x}^{y} g(a)\nabla a,$

ii) $\int_{x}^{y} \alpha f(a)\nabla a = \alpha \int_{x}^{y} f(a)\nabla a,$

iii) $\int_{x}^{y} f(a)\nabla a = -\int_{y}^{x} f(a)\nabla a;$

iv) $\int_{x}^{y} f(a)\nabla a = \int_{x}^{z} f(a)\nabla a + \int_{z}^{y} f(a)\nabla a;$

v) $\int_{x}^{y} f(\rho(a))g^{\nabla}(a)\nabla a = (fg)(y) - (fg)(x) - \int_{x}^{y} f^{\nabla}(a)g(a)\nabla a;$

vi) $\int_{x}^{y} f(a)g^{\nabla}(a)\nabla a = (fg)(y) - (fg)(x) - \int_{x}^{y} f^{\nabla}(a)g(\rho(a))\nabla a;$

vii) $\int_{x}^{x} f(a)\nabla a = 0.$

2. Main Result

Theorem 3.1 Let $\zeta,\vartheta,\eta,\gamma \in \mathrm{T}, \zeta < \vartheta$ and $g: [\zeta,\vartheta] \longrightarrow \mathbb{R}$ is differentiable and the parameter $\alpha \in [0,1]$ is given.

$$(1-\alpha)g(\gamma)+\frac{\lambda}{2}\big(g(\zeta)+g(\vartheta)\big)=\frac{1}{\vartheta-\zeta}\int_{\zeta}^{\vartheta} g^{\rho}(\eta)\nabla\eta+\frac{1}{\vartheta-\zeta}\int_{\zeta}^{\vartheta} p(\gamma,\eta)g^{\nabla}(\eta)\nabla\eta.$$

Then

$$p(\gamma,\eta)=\begin{cases}\eta-\left(\zeta+\alpha\dfrac{\vartheta-\zeta}{2}\right), & \zeta\le\eta<\gamma,\\ \eta-\left(\vartheta-\alpha\dfrac{\vartheta-\zeta}{2}\right), & \gamma\le\eta\le\vartheta.\end{cases}$$

Here, the following expression can be written using the property of the Nabla integral;

$$\int_{\zeta}^{\gamma} \eta-\left(\zeta+\alpha\frac{\vartheta-\zeta}{2}\right)g^{\nabla}(\eta)\nabla\eta=\left(\gamma-\left(\zeta+\alpha\frac{\vartheta-\zeta}{2}\right)\right)g(\gamma)+\alpha\frac{\vartheta-\zeta}{2}g(\zeta)-\int_{\zeta}^{\gamma} g^{\rho}(\gamma)\nabla\eta.$$

$$\int_{\gamma}^{\vartheta} \eta-\left(\vartheta-\alpha\frac{\vartheta-\zeta}{2}\right)g^{\nabla}(\eta)\nabla\eta=-\left(\gamma-\left(\vartheta-\alpha\frac{\vartheta-\zeta}{2}\right)\right)g(\gamma)+\alpha\frac{\vartheta-\zeta}{2}g(\vartheta)$$
$$-\int_{\gamma}^{\vartheta} g^{\rho}(\gamma)\nabla\eta.$$

Hereby,

$$\frac{1}{\vartheta-\zeta}\int_{\zeta}^{\vartheta} g^{\rho}(\eta)\nabla\eta+\frac{1}{\vartheta-\zeta}\int_{\zeta}^{\vartheta} p(\gamma,\eta)g^{\nabla}(\eta)\nabla\eta$$
$$=\frac{1}{\vartheta-\zeta}\int_{\zeta}^{\vartheta} g^{\rho}(\eta)\nabla\eta$$
$$+\frac{1}{\vartheta-\zeta}\left[(\vartheta-\zeta)(1-\alpha)g(\gamma)+\alpha\frac{\vartheta-\zeta}{2}\big(g(\zeta)+g(\vartheta)\big)-\int_{\gamma}^{\vartheta} g^{\rho}(\gamma)\nabla\eta\right]$$
$$=(1-\alpha)g(\gamma)+\frac{\lambda}{2}\big(g(\zeta)+g(\vartheta)\big).$$

Theorem 3.2 Let $\zeta, \vartheta, \eta,\ \gamma \in \mathrm{T}, \zeta < \vartheta$ and $g: [\zeta, \vartheta] \longrightarrow \mathbb{R}$ be nabla differentiable and the parameter $\alpha \in [0,1]$ is given. From here;

$$\left|(1-\alpha)g(\gamma) + \frac{\alpha}{2}\big(g(\zeta) + g(\vartheta)\big) - \frac{1}{\vartheta - \zeta}\int_{\zeta}^{\vartheta} g^{\rho}(\eta)\nabla\eta\right| \leq \frac{\mathrm{K}}{\vartheta - \zeta}\big(f_2(\zeta,\gamma) + f_2(\gamma,\vartheta)\big).$$

$$K = \begin{matrix} inf \\ \zeta < \gamma < \vartheta \end{matrix} |g^{\nabla}(\gamma)|$$

This inequality is absolute in the sense that the left side cannot be replaced by a larger side.

Using the previously accepted Theorem 3.1 with $p(\gamma, \eta)$ the following expression can be written.

$$\left|(1-\alpha)g(\gamma) + \frac{\alpha}{2}\big(g(\zeta) + g(\vartheta)\big) - \frac{1}{\vartheta - \zeta}\int_{\zeta}^{\vartheta} g^{\rho}(\eta)\nabla\eta\right| = \left|\frac{1}{\vartheta - \zeta}\int_{\zeta}^{\vartheta} p(\gamma,\eta)g^{\rho}(\eta)\nabla\eta\right|$$

$$\leq \frac{\mathrm{K}}{\vartheta - \zeta}\left(\int_{\zeta}^{\gamma}\left|\eta - \left(\zeta + \alpha\frac{\vartheta - \zeta}{2}\right)\right|\nabla\eta + \int_{\gamma}^{\vartheta}\left|\eta - \left(\vartheta - \alpha\frac{\vartheta - \zeta}{2}\right)\right|\nabla\eta\right)$$

$$= \frac{\mathrm{K}}{\vartheta - \zeta}\left[\int_{\zeta}^{\gamma}\left(\eta - \left(\zeta + \alpha\frac{\vartheta - \zeta}{2}\right)\right)\nabla\eta + \int_{\gamma}^{\vartheta}\left(\left(\vartheta - \alpha\frac{\vartheta - \zeta}{2}\right) - \eta\right)\nabla\eta\right]$$

$$= \frac{\mathrm{K}}{\vartheta - \zeta}\big(f_2(\zeta,\gamma) + f_2(\gamma,\vartheta)\big).$$

Corollary 3.3 Let's assume that the statement Theorem 3.2 is true. Then this inequality can be written

$$\left|g(\gamma) - \frac{1}{\vartheta - \zeta}\int_{\zeta}^{\vartheta} g^{\rho}(\eta)\nabla\eta\right| = \left|\frac{1}{\vartheta - \zeta}\int_{\zeta}^{\vartheta} p(\gamma,\eta)g^{\rho}(\eta)\nabla\eta\right|$$

$$\leq \frac{K}{\vartheta - \zeta}\left(\int_{\zeta}^{\gamma}|\eta - \zeta|\nabla\eta + \int_{\gamma}^{\vartheta}|\eta - \vartheta|\,\nabla\eta\right)$$

$$= \frac{K}{\vartheta - \zeta}\left(\int_{\zeta}^{\gamma}|\eta - \zeta|\nabla\tau + \int_{\gamma}^{\vartheta}|\vartheta - \eta|\,\nabla\tau\right) = \frac{K}{\vartheta - \zeta}\big(f_2(\zeta,\gamma) + f_2(\gamma,\vartheta)\big),$$

where $p(\gamma,\zeta) = 0$, $g(\gamma) = \gamma$, $\zeta =$, $\vartheta = \mathrm{T}_2$ and $\gamma = \mathrm{T}_2$ are taken into account in Theorem 3.2 to prove that the highest value reaching the smallest value in M is less than ℓ.

We obtain the expressions $g^{\nabla}(\gamma) = 1$ and $K = 1$. If we look at the left side of Theorem 2.2;

$$\left|g(\gamma) - \frac{1}{\vartheta-\zeta}\int_{\zeta}^{\vartheta} g^{\rho}(\eta)\nabla\eta\right| = \left|\mathrm{T}_2 - \frac{1}{\mathrm{T}_2-\mathrm{T}_1}\int_{\mathrm{T}_1}^{\mathrm{T}_2} \rho(\eta)\nabla\eta\right|$$
$$= \left|\mathrm{T}_2 - \frac{1}{\mathrm{T}_2-\mathrm{T}_1}\left(\int_{\mathrm{T}_1}^{\mathrm{T}_2} (\rho(\eta)+\eta)\nabla\eta - \int_{\mathrm{T}_1}^{\mathrm{T}_2} \eta\nabla\eta\right)\right|$$
$$= \left|\mathrm{T}_2 - \frac{1}{\mathrm{T}_2-\mathrm{T}_1}\left(\int_{\mathrm{T}_1}^{\mathrm{T}_2} (\eta^2)^{\nabla}\nabla\eta - \int_{\mathrm{T}_1}^{\mathrm{T}_2} \eta\nabla\eta\right)\right| = \left|-\mathrm{T}_1 - \frac{1}{\mathrm{T}_2-\mathrm{T}_1}\int_{\mathrm{T}_1}^{\mathrm{T}_2} \eta\nabla\eta\right|.$$

If we look from the right side of Theorem 3.2;

$$\frac{\mathrm{K}}{\vartheta-\zeta}\big(f_2(\zeta,\gamma) + f_2(\gamma,\vartheta)\big) = \frac{1}{\mathrm{T}_2-\mathrm{T}_1}\left(\int_{\mathrm{T}_1}^{\mathrm{T}_2} (\eta-\mathrm{T}_1)\nabla\eta - \int_{\mathrm{T}_1}^{\mathrm{T}_2} (\eta-\mathrm{T}_2)\nabla\eta\right)$$
$$= \frac{1}{\mathrm{T}_2-\mathrm{T}_1}\left(-\mathrm{T}_1\mathrm{T}_2 + {\mathrm{T}_1}^2 + \int_{\mathrm{T}_1}^{\mathrm{T}_2} \eta\nabla\eta\right) = -\mathrm{T}_1 + \frac{1}{\mathrm{T}_2-\mathrm{T}_1}\int_{\mathrm{T}_1}^{\mathrm{T}_2} \eta\nabla\eta.$$

So in this case;

$$\left|g(\gamma) - \frac{1}{\vartheta-\zeta}\int_{\zeta}^{\vartheta} g^{\rho}(\eta)\nabla\eta\right| \geq \frac{\mathrm{K}}{\vartheta-\zeta}\big(f_2(\zeta,\gamma) + f_2(\gamma,\vartheta)\big)$$

and

$$\left|g(\gamma) - \frac{1}{\vartheta-\zeta}\int_{\zeta}^{\vartheta} g^{\rho}(\eta)\nabla\eta\right| \leq \frac{\mathrm{K}}{\vartheta-\zeta}\big(f_2(\zeta,\gamma) + f_2(\gamma,\vartheta)\big).$$

With these expressions, the clarity of Ostrowski's inequality was demonstrated.

Conclusion

As a result of this study, we have adapted Ostrowski-type inequalities to Nabla derivative and integral. Delta derivative and integral forms were used in previous studies, and this study provides wider study areas and a different perspective on Ostrowski-type inequalities.

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