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Research Article

Sequences Generated by the lattice path problem with various vector sets

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Abstract

The lattice path problem involves finding a path between two specific points in space using only certain predefined vectors. The goal is to establish a relationship between the number of lattice paths to a point and the emergence of specific number sequences. This was achieved by analyzing lattice paths in a table within a cartesian coordinate system. The number of paths to a particular cell, starting from the first column of the table, was computed, and the results were analyzed through a computer program. This method revealed Fibonacci, Pell, Pell-Lucas, and Tribonacci sequences. Upon examining tables in dimensions higher than two, it was observed that the numbers found corresponded to the products of these special number sequences. Recursive relations were derived for the vector sets used in this process, and through these relations, identities among the number sequences were established.

Keywords: Lattice path , *Fibonacci sequence*, Pell-Lucas numbers, Higher-dimensional tables

MSC: 05A15, 11B83, 11B39, 05A18

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1. INTRODUCTION

Since the inception of mathematics, humans have continuously explored relationships and patterns among numbers, seeking to identify practical applications for them. This effort has resulted in numerous equations, special number sequences and various mathematical fields. Extensive research has been carried out to better understand and advance these areas. Among the most well-known number sequences that have emerged from this effort are the Fibonacci, Pell, and Lucas sequences. What distinguishes these sequences from many other number patterns is their inherent structure, which not only links their terms but also allows them to appear frequently in various real-world contexts. The widespread occurrence of these sequences in seemingly unrelated fields has motivated researchers to find potential applications in numerous domains. Due to the absence of a specific, guaranteed field for their findings, much of the research on these sequences has permeated various branches of literature. This very characteristic highlights the significance of special number sequences as fundamental tools in mathematics, linking diverse topics and demonstrating their importance in explaining the harmony of mathematical structures.

A lattice path is a path between two specific points, formed using only a predetermined set of vectors. Research on lattice paths has been conducted for many years, with particular focus intensifying in the 19th century and continuing into recent years. One notable study, presented in [5], calculates the number of distinct paths from the origin to a point (a,b)(a,b)(a,b) using the vectors {(1,0),(0,1),(1,1)},{(1,0), (0,1), (1,1)}{(1,0),(0,1),(1,1)}, minimizing the number of vectors used.

In the literature, most studies on lattice paths focus on two-dimensional lattice paths constructed using restricted sets of vectors. Among these, [1] has been particularly influential in inspiring this work. In this study, the Fibonacci, Pell, and Pell-Lucas sequences were derived using the vector set {(1,0),(1,1),(1,−1)}{(1,0), (1,1), (1,-1)}{(1,0),(1,1),(1,−1)} in two dimensions. Further exploration in this work extended the study to higher-dimensional spaces, where it was found that the relationship between the Pell and Pell-Lucas numbers becomes even more pronounced. Significant connections and recursive relations between these sequences were discovered through the analysis of lattice paths in higher-dimensional spaces using various vector sets.

This work originated during the exploration of lattice paths within a table. As we computed the number of distinct paths between selected points using specific vector sets, we discovered a connection between the resulting numbers. These findings led to the formulation of the following hypotheses.

2. MATERIALS and METHODS

In the overall structure of the paper, a path is traced from the first column of an m×n table to a specific cell within the table (the concept of columns changes when transitioning to higher-dimensional space). The total number of distinct paths leading to each cell is recorded inside that cell. It is assumed that there is exactly one path to each cell in the first column. In some parts of the paper, it was also assumed that all the cells in the first two columns have one path each.To briefly discuss the work that inspired this paper, [1] used the vectors {(1,0),(1,1),(1,-1)} 3n table to derive the Pell and Pell-Lucas sequences, and in a 4n table, the Fibonacci numbers.

The algorithm for this study follows a structured approach: first, a literature review is conducted and relevant publications are examined, culminating in the formulation of hypotheses. Next, Python software is developed to generate numbers, and it is checked whether these numbers correspond to a special number sequence. If the numbers align with a known sequence, recursive relations for the generated numbers are derived. These recursive relations are then used to identify identities associated with the special number sequences. Finally, the derived identities are proved using the relations of these special number sequences. This methodology allows for systematic exploration and validation of connections between lattice paths and special number sequences.

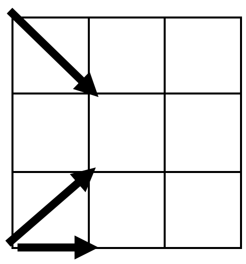
3. RESULTS

In this paper, we present the vector sets utilized and the corresponding sequences generated through the lattice path problem. The following vector sets were employed to explore the relationships between lattice paths and well-known number sequences:

1. **Vector Set** $\{(\mathbf{1},\mathbf{0}),(\mathbf{1},\mathbf{1})\}$
   This set was used to generate sequences that follow the **Fibonacci sequence**. The Fibonacci numbers are obtained by tracing paths from the origin to various points on the lattice, moving either horizontally or diagonally. The recursive relation for the Fibonacci sequence is $F_n = F_{n-1}$+F_{n-2} with initial conditions$F_0 = 0\ and\ F_1 = 1$
2. **Vector Set {(1,0),(1,1),(1,−1)}**
   This set was applied to derive the **Pell sequence**. The Pell numbers follow the recurrence relation $P_n = 2P_{n-1}$+P_{n-2} starting with $P_0 = 0\ and\ P_1 = 1$. It also led to the generation of the **Pell-Lucas sequence**, which is similar but differs in its initial conditions.
3. **Vector Set {(1,0),(0,1),(1,1)}**

   This set generates the Tribonacci sequence, where each term is the sum of the previous three terms. The recursive relation is $T_n = T_{n-1}$+T_{n-2} + T_{n-3}, with initial conditions $T_0 = 0$, $T_1 = T_2$ =1

In our study, we developed theories based on different vector groups.



vector set of
{(1,0),(1,1),(1,-1)}

1	2		5	12	29	70	169
1	3		7	17	41	99	239

1	2		5	12	29	70	169

Table 1. Sequences of 3n grid

1	2	5	13	34	89	233	F_{2n-1}
1	3	8	21	55	144	377	F_{2n}
1	2	5	13	34	89	233	

Table 2. Sequences of 4n grid

Let's create applications of this problem using different vector sets.

3.1 Some Vector Sets in 2 Dimensions

3.1.1 S = {(1,0), (0,1), (1,1), (1,-1)}.

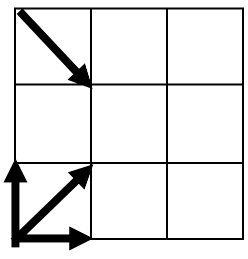
Let vector set be **S = {(1,0), (0,1), (1,1), (1,-1)}**. In this case, we will calculate the number of distinct paths on a 2D grid using these vectors. Each vector represents a possible step:

(1,0): A step to the right along the x-axis.

(0,1): A step upwards along the y-axis.

(1,1): A diagonal step in both the x and y directions.

(1,-1): A diagonal step moving right along the x-axis and down along the y-axis



We calculate the number of distinct path for 3 row grid.(3n grid)

The following data was obtained using a computer program. The program created the table by utilizing the recursive relationships between the numbers.

1	2	3	4	5	6	7	8
1	7	33	143	609	2583	10945	46367
1	5	21	89	377	1597	6765	28657
1	2	7	28	117	494	2091	8856

Table 3. Number of 3n grid for S

It was observed that there are number sequences

Top row: 1, 7, 33, 143, … → $F_{3s} - 1$.

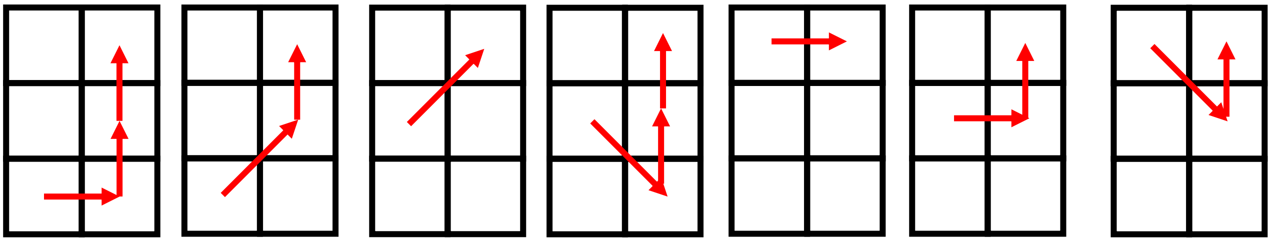
Middle row: 1, 5, 21, 89 … → F_{3s-1}

Bottom row: 1, 2, 7, 28, … → $\frac{1}{2}(F_{3s-1} + 1)$

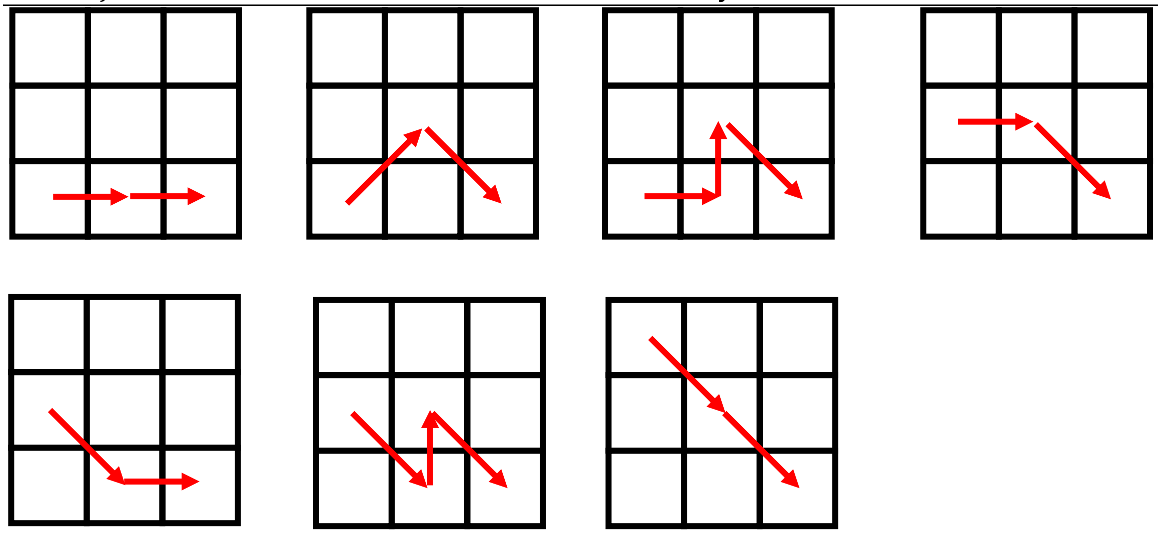
Note 1. The up vector is not used in the first column. The reason is that it is assumed that there is already 1 way to go to those compartments. Below, the paths to points (2,3) and (3,1) are modeled in a 3n table. As seen in the table, there are seven paths to these points.

Example 1.

The paths to points (2,3) as



The paths to points (3,1) as



Recursive relations,

Top row: $a_{3,n}$ →1, 7, 33, 143, 609 …

Middle row: $a_{2,n}$ →1, 5, 21, 89, 377 …

Bottom row: $a_{1,n}$ →1, 2, 7, 28, 117, 494 …

Recursive for bottom row,

$$a_{1,n} = a_{2,n-1} + a_{1,n-1}$$

In this expression, $a_{1,n-1}$ is replaced with the equivalent.

$$a_{1,n} = a_{2,n-1} + a_{2,n-2} + a_{1,n-2} = a_{2,n-1} + a_{2,n-2} + a_{2,n-3} + a_{1,n-3} =$$

The equivalent equations can be written in a similar way.

$$a_{1,n} = a_{1,1} + \sum_{k=1}^{n-1} a_{2,k}$$

Note: $a_{1,1} = 1$.

Recursive for top row,

$$a_{3,n} = a_{2,n} + a_{2,n-1} + a_{3,n-1}$$

In this expression, $a_{3,n-1}$ is replaced with the equivalent.

$$a_{3,n} = a_{2,n} + a_{2,n-1} + a_{2,n-1} + a_{2,n-2} + a_{3,n-2}$$

$$a_{3,n} = a_{2,n} + a_{2,n-1} + a_{2,n-1} + a_{2,n-2} + a_{2,n-2} + a_{2,n-3} + a_{3,n-3}$$

$$= a_{2,n} + a_{3,1} + a_{2,1} + 2\sum_{k=2}^{n-1} a_{2,k} = a_{2,n} + 2\sum_{k=1}^{n-1} a_{2,k}$$

Note: $a_{3,1}$=$a_{2,1}$=1

Recursive for to row,

$$a_{2,n} = a_{1,n} + a_{1,n-1} + a_{3,n-1} + a_{2,n-1}$$

We obtain using the relation

Based on the relations $a_{3,n}$ ve $a_{1,n}$ we found for we arrive at the following:

$$a_{2,n} = a_{1,n} + a_{1,n-1} + a_{3,n-1} + a_{2,n-1}$$

$$= 1 + \left[\sum_{k=1}^{n-1} a_{2,k}\right] + 1 + \left[\sum_{k=1}^{n-2} a_{2,k}\right] + a_{2,n-1} + 2\left[\sum_{k=1}^{n-2} a_{2,k}\right] + a_{2,n-1}$$

$$= 2 + 3a_{2,n-1} + 4\sum_{k=1}^{n-2} a_{2,k}$$

Corollary 1.

It was observed that the pattern $a_{2,n}$is the pattern F_{3n-1}. From this, for n≥3

$$F_{3n-1} = 2 + 3F_{3n-4} + 4\sum_{k=1}^{n-2} F_{3k-1}.$$

Proof.

The proof will be done by induction. The statement is true for n=3.

Let the statement be true for p,

$$F_{3p-1} = 2 + 3F_{3p-4} + 4\sum_{k=1}^{p-2} F_{3k-1}$$

The statement will be shown to be true for p+1 as well. Let n be assigned as p+1.

$$F_{3p+2} = 2 + 3F_{3p-1} + 4\sum_{k=1}^{p-1} F_{3k-1} = 2 + 3F_{3p-1} + 4\left(\sum_{k=1}^{p-2} F_{3k-1}\right) + 4F_{3p-4}$$
$$= 2 + +3F_{3p-4} + 4\left(\sum_{k=1}^{p-2} F_{3k-1}\right) + 3F_{3p-1} + F_{3p-4} = 4F_{3p-1} + F_{3p-4}$$

Let 3p=s,

$$F_{s+2} = F_{s+1} + F_s = 2F_s + F_{s-1} = 3F_{s-1} + 2F_{s-2} = 3F_{s-1} + F_{s-2} + F_{s-3} + F_{s-4}$$
$$= 4F_{s-1} + F_{s-4}.$$

Definition 1.

$I_m(n, V)$ represents the sum of the elements in the n-th column of a table constructed from the vector set V

For example $V = \{(1,0), (1,1), (1,-1)\}$ for $I_3(6, V)$=70+99+70=239.

1	2	5	12	29	70
1	3	7	17	41	99
1	2	5	12	29	70

The numbers are determined by a recursive relation.

Definiton 2. $C(s, k)$ represents the number of paths from the first column to the (s,k) cell. For example, in the table above, C(3,2)=7

Teorem 1.

∀n∊ N , n >1, $V_1 = \{(1,0), (1,1), (1,-1), (0,1)\}$ and $V_2 = \{(1,0), (1,1), (1,-1)\}$

$$I_m(n, V_1) > \sum_{k=0}^{n-1} I_{m-k}(n, V_2)(n-1)^k.$$

Proof.

Let's consider $I_{m-k}(n, V_2)$

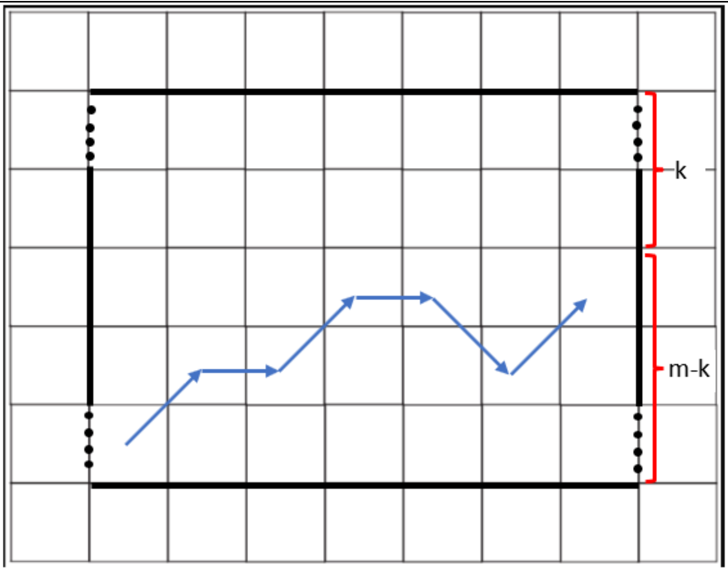


Table 4. Path of $I_m(n,V)$

If we add k (0,1) vectors to any part of the path in this way, the path will not leave the m*n table. This is because the path is at least as far away from the top row (m-k row) as k (0,1) vectors. Each of these k vectors can be placed in n-1 different places (not in the first column). However, we do not count all the paths to the nth column with V_2vectors. With this algorithm, the paths using k (0,1) vectors start from the lowest (1,k+1) cell vertically, that is, all the paths using k (0,1) vectors are not counted. For this reason,

$$I_m(n,V_1) > \sum_{k=0}^{n-1} I_{m-k}(n,V_2)(n-1)^k$$

inequality is ensured.

Let us support this theorem with an example.

Example 2. Let us m =n= 4

$$I_4(4,V_1) > \sum_{k=0}^{3} I_{4-k}(4,V_2)3^k$$

Let's find $I_4(4,V_1)$ A table can be created by utilizing the recursive relation.

1	10	63	341
1	8	45	233
1	5	22	103
1	2	7	29

$I_4(4, V_1)$=29+103+233+341=706

The values of '$I_4(4, V_2)$, $I_3(4, V_2)$, $I_2(4, V_2)$, $I_1(4, V_2)'$ are also required to be found.

1	2	5	13
1	3	8	21
1	3	8	21
1	2	5	13

$I_4(4, V_2)$=13+21+21+13=68

1	2	5	12
1	3	7	17
1	2	5	12

$I_3(4, V_2)$=12+17+12=41

1	2	4	8
1	2	4	8

$I_2(4, V_2)$=8+8=16

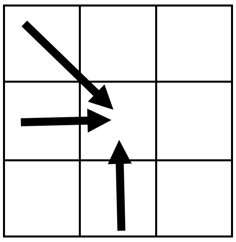
$I_1(4, V_2)$=1'dir.

$$706 > 68 * 3^0 + 41 * 3^1 + 16 * 3^2 + 1 * 3^3 = 362$$

Thus, the inequality has been established.

3.1.2. S = {(1,0), (0,1), (1,-1)}.

Let vector set be **S = {(1,0), (0,1), (1,-1)}**. In this case, we will calculate the number of distinct paths on a 2D grid using these vectors. Each vector represents a possible step:



For 3 row.

1	2	3	4	5	6	7	8
1	5	20	76	285	1065	3976	14840
1	4	15	56	209	780	2911	10864
1	2	6	21	77	286	1066	3977

Top row: 1 ,5 ,20, 76 … → https://oeis.org/A061278

Middle row: 1, 4, 15,56 … → https://oeis.org/A001353

Bottom row: 1, 2, 6, 21, 77 ,… → https://oeis.org/A101265

It has been identified that the number sequences are present in the OEIS

For 4 row:

1	2	3	4	5	6	7	8
1	7	36	168	756	3353	14783	65016
1	6	29	132	588	2597	11430	50233
1	4	16	67	288	1253	5480	24020
1	2	6	22	89	377	1630	7110

Third row from the bottom: 1, 6, 29,132, … → http://oeis.org/A112576 'The (2s+1)th terms of the row

Second row from the bottom : 1, 4 ,16,67, … → http://oeis.org/A112576' The (2s)th terms of the row

It has been identified that the number sequences are found in the OEIS..

3.2. Multidimensional Lattice Paths

Multidimensional structures and tables are powerful tools for modeling and analyzing complex systems in various mathematical fields. These tables help to understand the relationships between multiple parameters and dimensions in a given system, particularly in **lattice paths**, **multidimensional arrays**, and **combinatorics**.

We have extended our lattice path analysis to multidimensional space. In a d-dimensional space, we consider our vector set,

$$V_d = \{(1, h_1, h_2, \ldots, h_{d-1}) :\ 1 \leq k \leq d-1, h_k \in \{-1, 0, 1\}\}$$

The table with dimensions (d-1)x m is referred to as a *multidimensional table* or multidimensional array, where $T^{d}_{m,n}$, n×m×m×…×m represents the total number of dimensions.

It is the whole that remains in the table containing a certain x

The A.th : $A = \{a_h\colon 1 \leq h \leq d-1\}$and , $T^{d}_{m,n}$'de $(x, a_1, a_2, \ldots a_{d-1})$

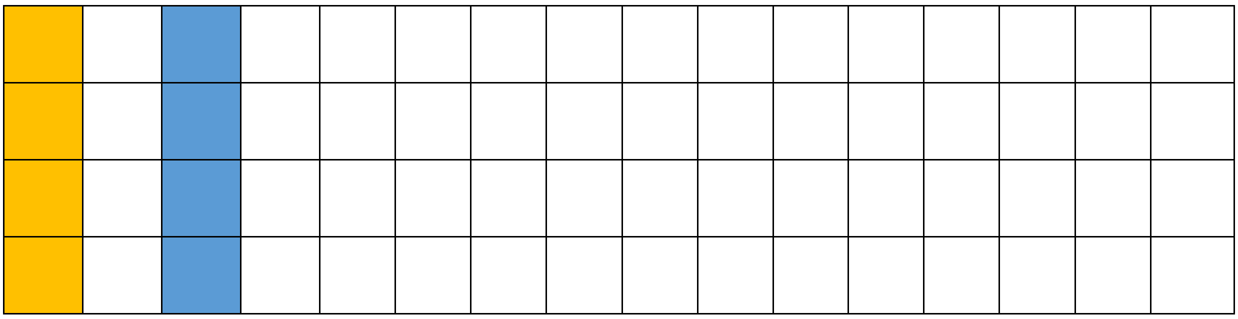
The set of cells in the form of {cells} where the x-axis is a variable and the other axes are fixed, can be considered as a slice or a row of a multidimensional structure.

y=4															
y=3															
y=2															
y=1															

$T^{2}_{4,16}$' is represented the third row

d-face: $T^{d}_{m,n}$, 1×m×m×…×m (d-1 times m). It is the whole that remains in the table containing a certain x.

It refers to all the cells in the table that have a specific x-value.



x=1	x=2	x=3	x=4												x=16

The column shown in yellow in $T^2_{5,16}$ is refer to 1. 2-face, The column shown in blue in $T^2_{5,16}$ is refer to 3. 2-face.

The number of paths reaching a desired compartment from the 1st d-face (x=1, others take values between 1 and m) within $T^d_{m,n}$ in the multi-dimensional space is calculated.

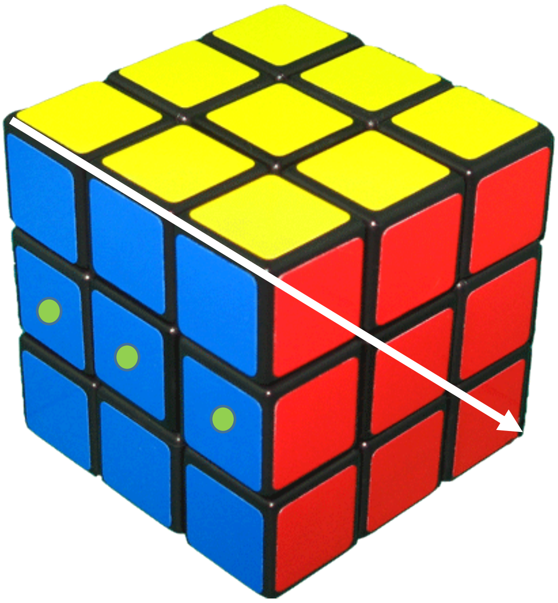


Figure 1. 3x3x3 Cube in 3D

The arrow drawn on the intelligence cube indicates the x-axis. The blue-faced cubes with green dots have the values y=2 and z=3, they are in the (2,3)-row in $T^3_{3,3}$The red-faced cubes represent the 3rd 3-face.

3.2. 1 Calculations 3x3x3 Cube in 3D

In this section, let's examine our theory on a cube shape, where each face contains a 3x3 grid of squares.

The paths in 3D $T^3_{3,n}$ were calculated by a computer program. The data is given below.

```
[1, 1, 1]  [4, 6, 4]  [25, 35, 25]  [144, 204, 144]  [841, 1189, 841]
[1, 1, 1]  [6, 9, 6]  [35, 49, 35]  [204, 289, 204]  [1189, 1681, 1189]
[1, 1, 1]  [4, 6, 4]  [25, 35, 25]  [144, 204, 144]  [841, 1189, 841]
```

x=1 (1.3-face) x=2(2.3-face) x=3 (3.3-face)

```
[4900, 6930, 4900] [28561, 40391, 28561] [166464, 235416, 166464]
[6930, 9801, 6930] [40391, 57121, 40391] [235416, 332929, 235416]
[4900, 6930, 4900] [28561, 40391, 28561] [166464, 235416, 166464]
```

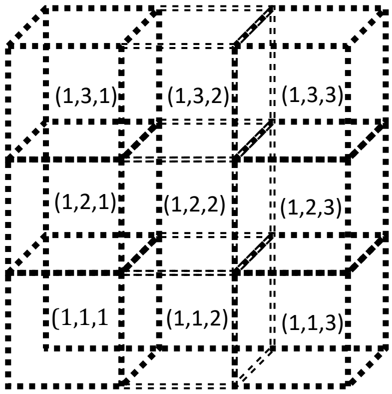
Example 3.

For example, let's find how many different ways to get to the (2,2,2) compartment in $T^3_{3,n}$,

Let's write the vectors in 3 dimensions first.

(1,0,0), (1,0,1), (1,0,-1), (1,1,0), (1,1,1), (1,1,-1), (1,-1,0), (1,-1,1), (1,-1,-1).

Now let's show the 1st 3-face:



1. 3-

The sections found on this page are:

(1,1,1), (1,1,2), (1,1,3), (1,2,1), (1,2,2), (1,2,3), (1,3,1), (1,3,2), (1,3,3)

The paths from these compartments to compartment (2,2,2) are:

Face	Vector	Reached Cell
(1,1,1)	(1,1,1)	(2,2,2)
(1,1,2)	(1,1,0)	(2,2,2)
(1,1,3)	(1,1,-1)	(2,2,2)
(1,2,1)	(1,0,1)	(2,2,2)
(1,2,2)	(1,0,0)	(2,2,2)
(1,2,3)	(1,0,-1)	(2,2,2)

(1,3,1)	(1,-1,1)	(2,2,2)
(1,3,2)	(1,-1,0)	(2,2,2)
(1,3,3)	(1,-1,-1)	(2,2,2)

There are 9 routes in total.

Let's color a face of a cube using recursive relations

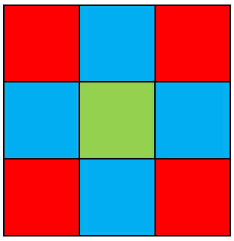


Figure 1. A face of Cube

In this section, we explore the theory applied to a cube shape, where each face is composed of a 3x3 grid of squares. The cube's faces are divided into different colored cells, and the analysis investigates the symmetry and relationships between these colored cells.

Symmetry of the Cube Faces

The top view of a 3-face cube shows the arrangement of cells in rows and columns. Due to the inherent symmetry in the vector set, the number of paths to each colored cell remains constant across different cells of the same color.

- Red Cells: These are located in the following rows: (1,1)-row, (1,3)-row, (3,1)-row, and (3,3)-row.
- Blue Cells: These are located in the (2,1)-row, (1,2)-row, (3,2)-row, and (2,3)-row.
- Green Cells: These are located in the (2,2)-row, representing the central position on the grid.

This symmetrical arrangement suggests a balance in the number of paths to each colored cell.

Pell numbers are denoted by P_n. Part of this sequence is as follows: 0, 1, 2, 5, 12, 29, 70, 169…

Half of the Pell-Lucas numbers are denoted by Q_n

The Pell-Lucas sequence of numbers is $a_0 = 2, a_1 = 2$ and n≥2 için $a_n = 2a_{n-1} + a_{n-2}$

The sequences Q_n is $q_0 = 1, q_1 = 1$ ve n≥2 için $q_n = 2q_{n-1} + q_{n-2}$

Part of this sequence is as follows: 1, 1, 3, 7, 17, 41, 99, 239…

N	0	1	2	3	4	5	6
Pell-Lucas	2	2	6	14	34	82	198
Q_n	1	1	3	7	17	41	99

We discovered the relationships of the colored cells in Figure 1 with some number sequences.

Red cell numbers:1, 4, 25, 144 … → ${P_n}^2$

Blue cell numbers: 1 6 35 204 … → P_nQ_n

Green cell numbers: 1 9 49 … → ${Q_n}^2$

Recursive form of cells.

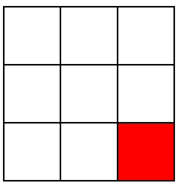
Red cell numbers → $a_{1,n}$

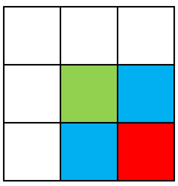
Blue cell numbers → $a_{2,n}$

Green cell numbers: → $a_{3,n}$

Let give information about basic recursive relations:

For $a_{1,n}$



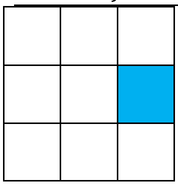


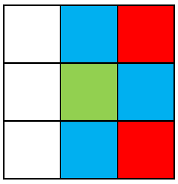
(n-1). 3-face

A red cell in n.3-face can be reached from (n-1)rd 3-face only by 2 blue, 1 red and 1 green cells

$$a_{1,n} = a_{1,n-1} + 2a_{2,n-1} + a_{3,n-1}$$

For $a_{2,n}$



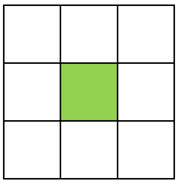


(n-1). 3-face

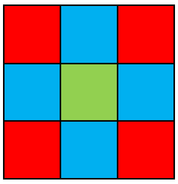
A blue cell in the nth 3rd face can be reached from only 3 blue, 2 red and 1 green cells in the (n-1) 3rd face.n

$$a_{2,n} = 2a_{1,n-1} + 3a_{2,n-1} + a_{3,n-1}$$

$a_{3,n}$ için:



n. 3-face



(n-1). 3-face

A green cell in the nth 3rd face can be reached from all cells in the (n-1) 3rd face.

$$a_{3,n} = 4a_{1,n-1} + 4a_{2,n-1} + a_{3,n-1}$$

$$a_{1,n} = a_{1,n-1} + 2a_{2,n-1} + a_{3,n-1}$$

$$a_{2,n} = 2a_{1,n-1} + 3a_{2,n-1} + a_{3,n-1}$$

$$a_{3,n} = 4a_{1,n-1} + 4a_{2,n-1} + a_{3,n-1}$$

Recursive for red numbers:

$$a_{1,n} = a_{3,n-1} + 2a_{2,n-1} + a_{1,n-1} = a_{3,n-1} + 2a_{2,n-1} + a_{3,n-2} + 2a_{2,n-2} + a_{1,n-2}$$
$$= a_{1,1} + \sum_{k=1}^{n-1} 2a_{2,k} + a_{3,k} = 1 + \sum_{k=1}^{n-1} 2a_{2,k} + a_{3,k}$$

Note: $a_{3,1} = 1$

Recursive for green numbers:

$$a_{3,n} = 4a_{1,n-1} + 4a_{2,n-1} + a_{3,n-1} = 4a_{1,n-1} + 4a_{2,n-1} + 4a_{1,n-2} + 4a_{2,n-2} + a_{3,n-2}$$
$$= 1 + 4\sum_{k=1}^{n-1} a_{1,k} + a_{2,k}$$

Sequences of cells of different colors were obtained

Red numbers:1, 4, 25, 144 … → ${P_n}^2$

Blue numbers: 1, 6, 35, 204 … → $P_n\, Q_n$

Green numbers : 1, 9, 49 … → ${Q_n}^2$

Let's obtain recursive relations for colors.

Red numbers → $a_{1,n}$

Blue numbers → $a_{2,n}$

Green numbers → $a_{3,n}$

$$a_{1,n} = a_{1,n-1} + 2a_{2,n-1} + a_{3,n-1}$$
$$a_{2,n} = 2a_{1,n-1} + 3a_{2,n-1} + a_{3,n-1}$$
$$a_{3,n} = 4a_{1,n-1} + 4a_{2,n-1} + a_{3,n-1}$$

Recursive for red numbers:

$$a_{1,n} = a_{3,n-1} + 2a_{2,n-1} + a_{1,n-1} = a_{3,n-1} + 2a_{2,n-1} + a_{3,n-2} + 2a_{2,n-2} + a_{1,n-2}$$
$$= a_{1,1} + \sum_{k=1}^{n-1} 2a_{2,k} + a_{3,k} = 1 + \sum_{k=1}^{n-1} 2a_{2,k} + a_{3,k}$$

Not: $a_{3,1} = 1$

Recursive for green numbers:

$$a_{3,n} = 4a_{1,n-1} + 4a_{2,n-1} + a_{3,n-1} = 4a_{1,n-1} + 4a_{2,n-1} + 4a_{1,n-2} + 4a_{2,n-2} + a_{3,n-2}$$
$$= 1 + 4\sum_{k=1}^{n-1} a_{1,k} + a_{2,k}$$

Some identities have been obtained from recursive relations.

Lemma 1.

$$P_n^{\ 2} = P_{n-1}^{\ \ 2} + 2P_{n-1}Q_{n-1} + Q_{n-1}^{\ \ 2}$$

Proof.

It is sufficient to prove the identity $P_n = P_{n-1} + Q_{n-1}$

$Let\ be\ \alpha = 1 + \sqrt{2}$ ve $\beta = 1 - \sqrt{2}$.

$P_n = \frac{\alpha^n - \beta^n}{2\sqrt{2}}$ [2] and $Q_n = \frac{\alpha^n + \beta^n}{2}$ [3]

$$= \frac{\alpha^{n-1} - \beta^{n-1}}{2\sqrt{2}} + \frac{\alpha^{n-1} + \beta^{n-1}}{2} = \frac{\alpha^{n-1} - \beta^{n-1}}{2\sqrt{2}} + \frac{\sqrt{2}(\alpha^{n-1} + \beta^{n-1})}{2\sqrt{2}}$$
$$= \frac{(1+\sqrt{2})\alpha^{n-1} - (1-\sqrt{2})\beta^{n-1}}{2\sqrt{2}} = \frac{(\alpha)\alpha^{n-1} - (\beta)\beta^{n-1}}{2\sqrt{2}}.$$

Lemma 2.

$$2P_n\, Q_n + Q_n^{\ 2} = Q_nQ_{n+1}$$

Proof.

Since $2P_n\, Q_n + Q_n^{\ 2} = Q_n(2P_n + Q_n)$ it is sufficient to prove the identity $Q_{n+1} = 2P_n + Q_n$

$Let\ us\ \alpha = 1 + \sqrt{2}$ ve $\beta = 1 - \sqrt{2}$

$\frac{\alpha^{n+1} + \beta^{n+1}}{2} = 2\frac{\alpha^n - \beta^n}{2\sqrt{2}} + \frac{\alpha^n + \beta^n}{2} = \frac{\sqrt{2}(\alpha^n - \beta^n)}{2} + \frac{\alpha^n + \beta^n}{2} = \frac{(1+\sqrt{2})\alpha^n + (1-\sqrt{2})\beta^n}{2}$

is provided, the proof is complete.

Corollary 1.

$$\boldsymbol{P_n^{\ 2} = 1 + \sum_{k=1}^{n-1} 2P_k\, Q_k + Q_k^{\ 2} = 1 + \sum_{k=1}^{n-1} Q_kQ_{k+1}}$$

Proof.

We start with the following recursive relation

$$P_n^{\ 2} = P_{n-1}^{\ 2} + 2P_{n-1}Q_{n-1} + Q_{n-1}^{\ 2}$$

To proceed, we replace $P_{n-1}^{\ 2}$ with its own recursive expansion:

$$P_{n-2}^{\ 2} + 2P_{n-2}Q_{n-2} + Q_{n-2}^{\ 2}$$

Substituting this into the original equation, we obtain:

$$P_n^{\ 2} = (2P_{n-1}Q_{n-1} + Q_{n-1}^{\ 2}) + (2P_{n-2}Q_{n-2} + Q_{n-2}^{\ 2}) + P_{n-2}^{\ 2}$$

Next, we apply the identity for $P_{n-2}^{\ 2}$:

$$P_{n-2}^{\ 2} = P_{n-3}^{\ 2} + 2P_{n-3}Q_{n-3} + Q_{n-3}^{\ 2}$$

Substituting this into the equation, we obtain

$$P_n^{\ 2} = \left(2P_{n-1}Q_{n-1} + Q_{n-1}^{\ 2}\right) + \left(2P_{n-2}Q_{n-2} + Q_{n-2}^{\ 2}\right) + \left(2P_{n-3}Q_{n-3} + Q_{n-3}^{\ 2}\right) + P_{n-3}^{\ 2}$$

This process continues recursively, and eventually, we reach the base case $P_1^{\ 2}$ Thus, the general form becomes:

$$P_n^{\ 2} = 1 + \sum_{k=1}^{n-1} 2P_k\, Q_k + Q_k^{\ 2}$$

Now, we use the identity:

$$2P_k\, Q_k + Q_k^{\ 2} = Q_k Q_{k+1}$$

Substituting this into the summation, we obtain:

$$P_n^{\ 2} = 1 + \sum_{k=1}^{n-1} Q_k Q_{k+1}$$

Corollary 2.

$$\boldsymbol{Q_n^{\ 2} = 1 + 4\sum_{k=1}^{n-1} P_k^{\ 2} + P_k Q_k}$$

Proof.

We are given the recursive relation for Q_n as:

$$Q_n = 2P_{n-1} + Q_{n-1}$$

Squaring both sides, we obtain the expression for ${Q_n}^2$:

$${Q_n}^2 = (2P_{n-1} + Q_{n-1})^2$$

Expanding the right-hand side, we get:

$${Q_n}^2 = 4{P_{n-1}}^2 + 4P_{n-1}Q_{n-1} + {Q_{n-1}}^2$$

Next, we replace ${Q_{n-1}}^2$ using its recursive relation:

$${Q_{n-1}}^2 = 4{P_{n-1}}^2 + 4P_{n-2}Q_{n-2} + {Q_{n-2}}^2$$

Substituting this into the original equation for ${Q_n}^2$ we obtain:

$${Q_n}^2 = (4{P_{n-1}}^2 + 4P_{n-1}Q_{n-1}) + (4{P_{n-2}}^2 + 4P_{n-2}Q_{n-2}) + {Q_{n-2}}^2$$

At this point, we continue the substitution process, replacing ${Q_{n-2}}^2$ with its own recursive expression:

$${Q_{n-2}}^2 = 4{P_{n-2}}^2 + 4P_{n-3}Q_{n-3} + {Q_{n-3}}^2$$

This leads to

$${Q_n}^2 = (4{P_{n-1}}^2 + 4P_{n-1}Q_{n-1}) + (4{P_{n-2}}^2 + 4P_{n-2}Q_{n-2}) + (4{P_{n-2}}^2 + 4P_{n-3}Q_{n-3}) + {Q_{n-3}}^2$$

We can continue this process, replacing ${Q_{n-s}}^2$ with its recursive expansion until we reach ${Q_1}^2$ After continuing this process, we can write the final expression as:

$${Q_n}^2 = 1 + 4\sum_{k=1}^{n-1} {P_k}^2 + P_kQ_k$$

3.2. 3 Calculations 4x4x4 Cube in 3D

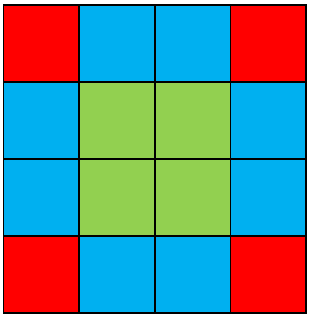
In this section, recursive relations and identities will be examined on a cube with a face consisting of a 4x4 grid of square cells.

3 dimensional $T_{4,n}^3$:

```
[1, 1, 1, 1]  [4, 6, 6, 4]  [25, 40, 40, 25]  [169, 273, 273, 169]
[1, 1, 1, 1]  [6, 9, 9, 6]  [40, 64, 64, 40]  [273, 441, 441, 273]
[1, 1, 1, 1]  [6, 9, 9, 6]  [40, 64, 64, 40]  [273, 441, 441, 273]
[1, 1, 1, 1]  [4, 6, 6, 4]  [25, 40, 40, 25]  [169, 273, 273, 169]
```

x=1 (1. 3-face)   x=2 (2. 3-face)   x=3 (3. 3-face)

```
[1156, 1870, 1870, 1156]  [7921, 12816, 12816, 7921]
[1870, 3025, 3025, 1870]  [12816, 20736, 20736, 12816]
[1870, 3025, 3025, 1870]  [12816, 20736, 20736, 12816]
[1156, 1870, 1870, 1156]  [7921, 12816, 12816, 7921]
```



$T_{4,n}^{3}$ in 3 dimensional

F_n is the n-th Fibonacci number.

Red numbers : 1 4 25 169 … → ${F_{2n-1}}^2$ This sequence is denoted as $a_{1,n}$

Blue numbers: 1 6 40 273 1870 … → $F_{2n-1}F_{2n}$ This sequence is denoted as $a_{2,n}$

Green numbers: 1 9 64 441 … → ${F_{2n}}^2$ This sequence is denoted as $a_{3,n}$.

Let's derive the fundamental recursive relations.

$$a_{1,n} = a_{1,n-1} + 2a_{2,n-1} + a_{3,n-1}$$

$$a_{2,n} = a_{1,n-1} + 3a_{2,n-1} + 2a_{3,n-1}$$

$$a_{3,n} = a_{1,n-1} + 4a_{2,n-1} + 4a_{3,n-1}$$

The recursive relations are as follows:

$$a_{3,1}=1$$

$$a_{3,n} = a_{1,n-1} + 4a_{2,n-1} + 4a_{3,n-1}$$

$$= a_{1,n-1} + 4a_{2,n-1} + 4\left(a_{1,n-2} + 4a_{2,n-2} + 4a_{3,n-2}\right)$$

$$= a_{1,n-1} + 4a_{2,n-1} + 4a_{1,n-2} + 16a_{2,n-2} + 16a_{3,n-2}$$

$$= a_{1,n-1} + 4a_{2,n-1} + 4a_{1,n-2} + 16a_{2,n-2} + 16\left(a_{1,n-3} + 4a_{2,n-3} + 4a_{3,n-3}\right)$$

$$= a_{1,n-1} + 4a_{2,n-1} + 4a_{1,n-2} + 16a_{2,n-2} + 16a_{1,n-3} + 64a_{2,n-3} + 64a_{3,n-3}$$

$$= 4^{n-1}a_{3,1} + \sum_{k=1}^{n-1} 4^{k-1}a_{1,n-k} + 4^k a_{2,n-k}$$

Corollary 3.

$$F_{2n}{}^2 = 4^{n-1} + \sum_{k=1}^{n-1} 4^{k-1}F_{2n-2k-1}{}^2 + 4^k F_{2n-2k-1}F_{2n-2k}$$

Proof.

We aim to prove the following identity using mathematical induction

First, let's verify the identity for the base case when p=2

$$F_{2p}{}^2 = 4^{p-1} + \sum_{k=1}^{p-1} 4^{k-1}F_{2p-2k-1}{}^2 + 4^k F_{2p-2k-1}F_{2p-2k}$$

$$F_{2.2}{}^2 = 4^{2-1} + \sum_{k=1}^{2-1} 4^{k-1}F_{2.2-2k-1}{}^2 + 4^k F_{22-2k-1}F_{2.2-2k}$$

This simplifies further as:

$F_4{}^2$=4+(4^0·$F_1{}^2 + 4^1)F_1F_2$=3^2=4+$(1 + 4)1.1$= 9

Thus, the base case holds.

Assume that the formula holds for some p. That is, assume the following:

$$F_{2p}{}^2 = 4^{p-1} + \sum_{k=1}^{p-1} 4^{k-1}F_{2p-2k-1}{}^2 + 4^k F_{2p-2k-1}F_{2p-2k}$$

Now, we need to prove that the formula holds for p+1. We begin with the following expression:

$$F_{2p+2}{}^2 = 4^p + \sum_{k=1}^{p} 4^{k-1}F_{2p-2k+1}{}^2 + 4^k F_{2p-2k+1}F_{2p-2k+2}$$

Now, shift the summation index by setting k=k+1, so the index range becomes from 0 to p−1.

$$F_{2p+2}{}^2 = 4^p + \sum_{k=0}^{p-1} 4^k F_{2p-2k-1}{}^2 + 4^{k+1} F_{2p-2k-1}F_{2p-2k}$$

Expanding the left-hand side:

$$F_{2p}{}^2 + 2F_{2p}F_{2p+1} + F_{2p+1}{}^2 = 4^p + \sum_{k=0}^{p-1} 4^k F_{2p-2k-1}{}^2 + 4^{k+1} F_{2p-2k-1}F_{2p-2k}$$

Now, subtract $F_{2p}{}^2$ from both sides:

$$2F_{2p}F_{2p+1} + F_{2p+1}{}^2 =$$

$$= 4^p - 4^{p-1} + \sum_{k=0}^{p-1} 4^k F_{2p-2k-1}{}^2 + 4^{k+1}F_{2p-2k-1}F_{2p-2k}$$

$$= 3 * 4^{p-1} + F_{2p-1}{}^2 + 4F_{2p-1}F_{2p} + 3\sum_{k=1}^{p-1} 4^{k-1}F_{2p-2k-1}{}^2 + 4^k F_{2p-2k-1}F_{2p-2k}$$

Now, subtract $F_{2p-1}{}^2 + 4F_{2p-1}F_{2p}$ from both sides

$$2F_{2p}F_{2p+1} + F_{2p+1}{}^2 - F_{2p-1}{}^2 - 4F_{2p-1}F_{2p} = 3(4^{p-1} + \sum_{k=1}^{p-1} 4^{k-1}F_{2p-2k-1}{}^2 + 4^k F_{2p-2k-1}F_{2p-2k})$$

Finally, simplify the right-hand side:

$3F_{2p}{}^2$. More clearly,

$$2F_{2p}F_{2p+1} + F_{2p+1}{}^2 - F_{2p-1}{}^2 - 4F_{2p-1}F_{2p} = 3F_{2p}{}^2$$

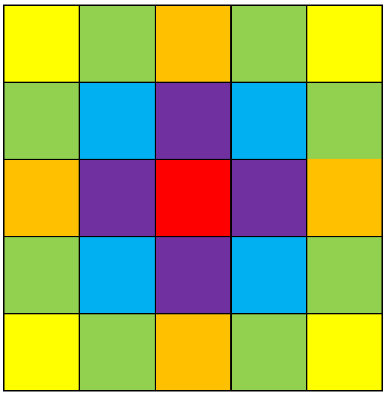
$$\begin{aligned} 2F_sF_{s+1} + F_{s+1}{}^2 - F_{s-1}{}^2 - 4F_{s-1}F_s &= 2F_sF_{s+1} + F_{s+1}{}^2 - F_{s-1}F_{s+1} - 3F_{s-1}F_s \\ &= 2F_sF_{s+1} + F_{s+1}{}^2 - F_{s-1}F_{s+2} - 2F_{s-1}F_s = 2F_s{}^2 + F_{s+1}{}^2 - F_{s-1}F_{s+2} \\ &= 2F_s{}^2 + F_{s+1}{}^2 - F_{s-1}F_{s+1} - F_{s-1}F_s = 2F_s{}^2 + F_sF_{s+1} - F_{s-1}F_s = 3F_s{}^2 \end{aligned}$$

This completes the inductive step, and hence the identity holds for p+1.

3.2. 4 Calculations 5x5x5 Cube in 3D

In this section, let's examine our theory on a cube shape, where each face contains a 5x5 grid of squares.

$T_{5,n}^{3}$ in 3 dimensional



Yellow numbers : 1, 4, 25, 169, 1225, 9025, 67081, 499849, 3728761, 27825625, 207676921, 1550075641

Green numbers: 1, 6, 40, 286, 2100, 15580, 116032, 865368, 6457264, 48192400, 359698560

Orange numbers: 1, 6, 45, 325, 2415, 17955, 133903, 998991, 7455591, 55645975, 415339431

Blue numbers: 1, 9, 64, 484, 3600, 26896, 200704, 1498176, 11182336, 83466496, 623001600

Purple number: 1, 9, 72, 550, 4140, 30996, 231616, 1729512, 12911184, 96375664

Red numbers: 1, 9, 81, 625, 4761, 35721, 267289, 1996569, 14907321, 111281401

It was seen that the number patterns found were in the form of products of number patterns in $T_{5,n}^{2}$ For example, red numbers in 3 dimensions are the square of red numbers in 2 dimensions. Green numbers in 3 dimensions are multiplied by a blue number and a yellow number in 2 dimensions.

The table formed by using the V_2 vector set in 2 dimensions

1	**2**	**5**	**13**	**35**	**95**	**259**	**707**	**1931**
1	**3**	**8**	**22**	**60**	**164**	**448**	**1224**	**3344**
1	**3**	**9**	**25**	**69**	**189**	**517**	**1413**	**3861**

1	**3**	**8**	**22**	**60**	**164**	**448**	**1224**	**3344**
1	**2**	**5**	**13**	**35**	**95**	**259**	**707**	**1931**

***Note*2** .The numbers in $T^3_{3,n}$ ve $T^3_{4,n}$T were also the product of their 2-dimensional states.

Note3. The numbers in $T^3_{m,n}$ ve $T^3_{m,n}$ were also the product of their 2-dimensional states.
Corollary4. In $T^d_{m,n}$, the (h_1, h_2, … , h_{d-1})-th row's s-th element is equal to the product of the s-th elements of the(h_1)-th row, (h_2)-th row, ..., and (h_{d-1})- th row in T(m,n) .

Proof.

Let's prove this for a three-dimensional table.

In $T^3_{m,n}$, to reach the cell((s, h_1, h_2) we can break the movement into two types: movement along the y-axis and movement along the z-axis.

Movement along the y-axis: Ignoring the z-axis, the movement is from the point (1,1,z) to (s, h_1, z)As we can see, this movement is two-dimensional. The number of paths between these two points corresponds to the s-th element of the (h_1)-th row in $T^2_{m,n}$

Movement along the z-axis: Ignoring the y-axis, the movement is from the point (1,y,1)to (s,y,h_2). Similarly, this is a two-dimensional movement. The number of paths between these points corresponds to the sss-th element of the (h_2)-th row in $T^2_{m,n}$

The movements along the y-axis and z-axis are independent events. Each vector in the vector set progresses the same distance along the x-axis (1 unit). The vector can move -1, 0, or 1 unit along the y-axis, but the choice of movement along the y-axis does not affect the movement along the z-axis. Similarly, the movement along the z-axis can also be -1, 0, or 1 unit.

A similar approach can be applied to a table in n dimensions. As the number of dimensions increases, the number of axes increases, and movement along each axis remains independent of the others. Therefore, the s-th element of the ($h_{1)}$-th row, (h_2)-th row, ..., and (h_{d-1})-th row in $T^2_{m,n}$ will be multiplied.

3.2.5 Algorithm software

Codes for Section 3.1

We used the following python program to calculate tables for

S = {(1,0), (0,1), (1,1), (1,-1)}.

```
from prettytable import PrettyTable


m=3 # rows
```

```
n=7 # columns


table_columns=[]

for h in range (0,n):

    table_columns.append(""+str(h))

table = PrettyTable(table_columns) # Create the table with columns
named from 1 to n


first_column=[1 for i in range(m)]

column_list=[first_column] # Initialize the column_list with the first
column being 1,1,1,...


# Generate other columns up to n

for a in range(1,n):

    precolumn=column_list[a-1] # previous column

    currcolumn=[precolumn[0]+precolumn[1]] # Initialize the current
column with the bottom element

    for i in range(1,m-1):

        currcolumn.append(currcolumn[i-1]+precolumn[i-
1]+precolumn[i]+precolumn[i+1]) # Apply the recursive formula

    currcolumn.append(currcolumn[m-2]+precolumn[m-2]+precolumn[m-1])

    column_list.append(currcolumn)


# Turn the column_list to a row list for creating the table.

row_list=[]

for i in range(m):

    currrow=[]

    for a in range(n):

        currrow.append(column_list[a][i])

    row_list.append(currrow)
```

```
for b in range(1,m+1):

    table.add_row(row_list[-b])

print(table)
```

The code for **S = {(1,0), (0,1), (1,-1)}** only requires a slight adjustment:

```
# Generate other columns up to n

for a in range(1,n):

    precolumn=column_list[a-1] # previous column

    currcolumn=[precolumn[0]+precolumn[1]] # Initialize the current
column with the bottom element

    for i in range(1,m-1):

        currcolumn.append(currcolumn[i-1]+precolumn[i]+precolumn[i+1])
# Apply the recursive formula

    currcolumn.append(currcolumn[m-2]+precolumn[m-1])

    column_list.append(currcolumn)
```

Code for Section 3.2

We used the following code to generate the faces of cubes.

```
x=3 # x

y=3 # y

z=5 # z or the index that faces will be generated up to


first_face=[]

row1=[]

# Create the first face that consists of only 1s.

for i in range(x):

    row1.append(1)

for i in range(y):
```

```
    first_face.append(row1)


presurface = first_face # Store the previous face, initially it is the
first face


# Generate other faces up to index z

for i in range(1,z):

    cursurface=[]


    # First row

frow=[presurface[0][0]+presurface[0][1]+presurface[1][0]+presurface[1][
1]]

    for b in range(1,x-1):

        frow.append(presurface[0][b]+presurface[0][b-
1]+presurface[0][b+1]+presurface[1][b]+presurface[1][b+1]+presurface[1]
[b-1]) # Recursive formula

    frow.append(presurface[0][x-1]+presurface[0][x-2]+presurface[1][x-
1]+presurface[1][x-2])

    cursurface.append(frow)

    print(frow)


     # Midle rows (not generated for y<=2)

    if y>2:

        for r in range(1,y-1):

            mrow=[]

            # Apply the recursive formula with separate formulas for
edges

            for b in range(x):

                if b==0:
```

```
mrow.append(presurface[r][0]+presurface[r][1]+presurface[r-
1][0]+presurface[r-1][1]+presurface[r+1][1]+presurface[r+1][0])
                elif b==x-1:
                    mrow.append(presurface[r][x-1]+presurface[r][x-
2]+presurface[r-1][x-1]+presurface[r-1][x-2]+presurface[r+1][x-
1]+presurface[r+1][x-2])
                else:
                    mrow.append(presurface[r][b]+presurface[r][b-
1]+presurface[r][b+1]+presurface[r-1][b]+presurface[r-
1][b+1]+presurface[r-1][b-
1]+presurface[r+1][b]+presurface[r+1][b+1]+presurface[r+1][b-1])
            print(mrow)
            cursurface.append(mrow)


    #Last row
    lrow=[presurface[y-1][0]+presurface[y-1][1]+presurface[y-
2][0]+presurface[y-2][1]]
    for b in range(1,x-1):
        lrow.append(presurface[y-1][b]+presurface[y-1][b-
1]+presurface[y-1][b+1]+presurface[y-2][b]+presurface[y-
2][b+1]+presurface[y-2][b-1])
    lrow.append(presurface[y-1][x-1]+presurface[y-1][x-2]+presurface[y-
2][x-1]+presurface[y-2][x-2])
    cursurface.append(lrow)
    print(lrow)


    print("") # Print empty line to separate faces
    presurface = cursurface # Update the previous face
```

4. DISCUSSION

In a two-dimensional table, the Fibonacci, Pell, and Pell-Lucas sequences and their products and powers have been obtained. By changing the vector set in the two-dimensional table, the Tetrabonacci sequence can be reached.

As shown in Corollary 4, the numbers in$T^d_{m,n}$ are products of the numbers in $T^2_{m,n}$. This result holds true when all vectors in a well-defined vector set advance the same unit along the x-axis (the important factor is not how many units they move, but that they move the same number of units). However, there are also well-defined vector sets where the x-axis is not a fixed number. Studies can be conducted to explore the relationship between the numbers in these vector sets in two-dimensional tables and multi-dimensional tables.

In these studies, matrices and linear algebra can be used to further investigate the connections and properties of these sequences.

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