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Research Article

Decay of solutions for a singular parabolic m-biharmonic equation with logarithmic nonlinearity

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Abstract
In this work, we obtain a singular parabolic m-biharmonic equation with logarithmic nonlinearity. We establish the decay of the solutions and then derive it using the potential well method and the Hardy-Sobolev inequality.

Keywords: ***Decay, logarithmic nonlinearity, singular potential.***

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1. INTRODUCTION

In this work, we investigate the following the initial-boundary value problem of a singular parabolic m-biharmonic equation with logarithmic nonlinearity

$$\begin{cases} \frac{z_t}{|x|^s} + \Delta\left(|\Delta|^{m-2}\Delta z\right) = |z|^{r-2} z \ln z, & (x,t) \in \Omega \times (0,T), \\ z(x,t) = \Delta z(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ z(x,0) = z_0(x) & x \in \Omega, \end{cases}$$

here $\Omega \subset R^n \quad (n > 2)$ be open bounded Lipschitz domain with a smooth boundary $\partial\Omega$ and $\max\left\{1, \frac{2n}{n+4}\right\} < m \leq r < m\left(1 + \frac{4}{n}\right)$, $0 \leq s \leq 2$ is a constant. Also, $T > 0$ and $x = (x_1, x_2, ..., x_n)$, $|x| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$.

The equations with m-biharmonic operators models physical phenomena in many fields, such as the traveling wave in suspension bridges [10], the theory of pseudoplastic non-Newtonian fluid [13], phase transformation [14].

In 2021, Han [8] proved the following the equation of the form

$$z_t + \Delta^2 z = k(t) f(z).$$

He established the explosion in finite time using differential inequalities. Furthermore, he derived both upper and lower limits for the time at which the explosion happens.

Han [7] studied the following the equation of the form

$$\frac{z_t}{|x|^2} - \Delta z = k(t)|z|^{p-1} z.$$

He proved the upper and lower bounds on the blow-up time of weak solutions.
In 2021, Thanh et al. [17] considered the reaction-diffussion parabolic problem with time dependent coefficients

$$\frac{z_t}{|x|^4} + \Delta^2 z = k(t)|z|^{p-1} z.$$

They proved the upper and lower bound for blow-up time. Problems with variable coefficients have been handled carefully in several papers, some results relating the local existence, global existence, blow up and stability have been found [4, 5, 6, 7, 15, 17].
In 2023, Wu et al. [19] investigated the following fourth-order parabolic equation

$$\frac{z_t}{|x|^4} + \Delta^2 z - \Delta z_t = |z|^{p-2} z \ln|z|.$$

They obtained finite-time blowup results of weak solutions using the Galerkin method and determined upper and lower bounds for the blowup time.
In 2020, Deng and Zhou [2] considered the following of singular and nonlinear parabolic equations with logarithmic source term

$$\frac{z_t}{|x|^s} + \Delta z = z \ln|z|.$$

They obtained infinite time blow-up of the solutions and the global existence.
In 2024, Yang [21] considered the following $p-$ Laplacian type pseudo-parabolic equation with singular potential and logarithmic nonlinearity

$$\frac{z_t}{|x|^s} + \Delta_p z - \Delta z_t = |z|^{q-2} z \ln|z|.$$

He has established a new criterion for solutions to blow up in finite time using Gagliardo-Nirenberg's interpolation inequality and inverse Sobolev inequality.
In [18], Thanh et al. proved the higher-order version $\Delta\left(|\Delta|^{m-2}\Delta\right)$ of the $p-$ Laplacian and the function $k(t)$ non- newtonian filtration equation and obtained the blow-up result with lower and upper bounded.
In 2023, Liu and Fang [12] investigated the following of singular parabolic p-biharmonic equation with logarithmic nonlinearity

$$\frac{z_t}{|x|^s} + \Delta\left(|\Delta|^{p-2}\Delta z\right) = |z|^{q-2} z \log|z|.$$

They obtained the global solvability, infinite and finite time blow-up phenomena and derive the upper bound of blow-up time as well as the estimate of blow-up rate. Furthermore, the results of blow-up with arbitrary initial energy and extinction phenomena are presented.
In 2024, Wu et al. [20] considere the following $p-$ Laplacian equation with singular potential and logarithmic nonlinearity

$$|x|^{-s} z_t - \Delta_p z = |z|^{q-2} z \log|z|.$$

They established the results of the decay and the blow-up of solutions with arbitrary initial energy and the conditions of extinction.
This work is organized as follows:

- In part 2, we give some assumptions needed in this work.
- In Part 3, we investigate decay of solutions by using the Komornik's inequality.

2. MATERIALS and METHODS

In this part, we present certain lemmas and assumptions required for the formulation and proof of our results. Let $\|.\|$, $\|.\|_p$ and $\|.\|_{W^{m,r}(\Omega)}$ indicate the typical $L^2(\Omega)$, $L^r(\Omega)$ and $W^{m,r}(\Omega)$ norms (see [1, 3]). By problem (5), assume that r and $g(.)$ satisfy the following conditions:
Multiplying equation (5) by z_t and integrating over $\Omega\times[0,t)$, we have

$$\int_0^t \left\| \frac{z_t}{|x|^{s/2}} \right\|^2 d\tau + \frac{1}{m}\|\Delta z\|_m^m$$
$$= \frac{1}{r}\int_\Omega |z|^r \ln z dx - \frac{1}{r}\int_\Omega |z_0|^r \ln|z_0| dx - \frac{1}{r^2}\|z\|_r^r + \frac{1}{r^2}\|z_0\|_r^r .$$

For each $z \in H_0^2(\Omega)\cap L^r(\Omega)$ and $t\in[0,\infty)$ define the functinals of the problem (5) following:

Energy functional is as follows:

$$J(z) = \frac{1}{m}\|\Delta z\|_m^m - \frac{1}{r}\int_\Omega |z|^r \ln z dx + \frac{1}{r^2}\|z\|_r^r ,$$

and Nehari functional is as follows:

$$I(z) = \|\Delta z\|_m^m - \int_\Omega |z|^r \ln z dx.$$

$$E(t) = \int_0^t \left\| \frac{z_\tau}{|x|^{s/2}} \right\|^2 d\tau + \frac{1}{m}\|\Delta z\|_m^m - \frac{1}{r}\int_\Omega |z|^r \ln z dx + \frac{1}{r^2}\|z\|_r^r .$$

Then it follows from (12) and (14) that

$$J(z) = \frac{1}{r}I(z) + \frac{r-m}{mr}\|\Delta z\|_m^m + \frac{1}{r^2}\|z\|_r^r .$$

Furthermore, we introduce the Nehari manifold$W$

$$\mathrm{N} = \left\{ z \in H_0^m(\Omega)\setminus\{0\} \ : \ I(z) = 0 \right\},$$

and the following sets:

$$\begin{aligned}
\mathrm{W}_1 &= \{ z \in H_0^m(\Omega)\setminus\{0\} : \ J(z) < d \}, \\
\mathrm{W}_1^+ &= \{ z \in \mathrm{W}_1 \ : \ I(z) > 0 \}, \\
\mathrm{W}_1^- &= \{ z \in \mathrm{W}_1 \ : \ I(z) < 0 \}, \\
\mathrm{W}_2 &= \{ z \in H_0^m(\Omega)\setminus\{0\} : \ J(z) = d \},
\end{aligned}$$

The depth of potential well is defined as follows:

$$d = \inf_{z\in N} J(z).$$

Now, we give some definitions.

Definition 1 ***(Weak solution)*** *A function* z *is called a weak solution to equation (5) if* $z \in L^{\infty}\left(0,T;H_0^2(\Omega)\cap L^r(\Omega)\right)$ *and* $\frac{z_t}{|x|^{s/2}} \in L^2\left(0,T;L^2(\Omega)\right)$ *where* z *satisfies the following equation:*

$$\left\langle \frac{z_t}{|x|^s},\varphi \right\rangle + \left\langle |\Delta z|^{m-2}\Delta z, \Delta\varphi \right\rangle = \left\langle |z|^{r-2} z\ln z,\varphi \right\rangle,$$

for all $\varphi \in H_0^2(\Omega)$ *and* $t\in[0,T)$.

After that, in Lemma 2, we outline some fundamental properties of the fiber mapping $J(\lambda z)$ that can be verified directly.

Lemma 2 *Assume that* $z \in H_0^m(\Omega)\setminus\{0\}$, *then*

(i) $\lim_{\lambda\to 0^+} J(\lambda z) = 0, \quad \lim_{\lambda\to+\infty} J(\lambda z) = -\infty.$

(ii) There exists a unique $\lambda^* = \lambda^*(z) > 0$ *so that* $\frac{d}{d\lambda} J(\lambda z)\big|_{\lambda=\lambda^*} = 0.$

(iii) $J(\lambda z)$ *is increasing on* $0<\lambda<+\infty$, *and attains the maximum at* $\lambda = \lambda^*$.

(iv) $I(\lambda z) > 0$ *for* $0<\lambda<\lambda^*$, $I(\lambda z)<0$ *for* $\lambda^*<\lambda<+\infty$, *and* $I(\lambda^* z) = 0.$

Proof

(i) By the definition of $J(u)$, we obtain

$$J(\lambda z) = \frac{1}{m}\lambda^m \|\Delta z\|_m^m - \frac{\lambda^r}{r}\ln\lambda \|z\|_r^r - \frac{\lambda^r}{r}\int_\Omega |z|^r \ln z dx + \frac{\lambda^r}{r^2}\|z\|_r^r,$$

where $\lambda>0$. Therefore, it is evident that the conclusion of (i) is valid

(ii) By differentiating $J(\lambda z)$ at λ we get:$\mathcal{A}$

$$\begin{aligned}\frac{d}{d\lambda}J(\lambda z) &= \lambda^{m-1}\|\Delta z\|_m^m - \lambda^{r-1}\ln\lambda\|z\|_r^r - \lambda^{r-1}\int_\Omega |z|^r\ln z dx \\ &= \lambda\left[\lambda^{m-2}\|\Delta z\|_m^m - \lambda^{r-2}\ln\lambda\|z\|_r^r - \lambda^{r-2}\int_\Omega |z|^r \ln z dx\right]\end{aligned}$$

Let $\mathrm{A}(\lambda z) = \frac{1}{\lambda}\frac{d}{d\lambda}J(\lambda z)$, then

$$\begin{aligned}\frac{d}{d\lambda}\mathrm{A}(\lambda z) &= -(r-2)\lambda^{r-3}\ln\lambda\|z\|_r^r - \lambda^{r-3}\|z\|_r^r - (r-2)\lambda^{r-3}\int_\Omega |z|^r \ln z dx \\ &= -\lambda^{r-3}\left[(r-2)\ln\lambda\|z\|_r^r - \|z\|_r^r + (r-2)\int_\Omega |z|^r\ln z dx\right]\end{aligned}$$

Hence, by taking

$$\lambda_1 = \exp\left[\frac{\|z\|_r^r + (r-2)\int_\Omega |z|^r \ln z dx}{(2-r)\|z\|_r^r}\right] > 0,$$

so that

$$\frac{d}{d\lambda}\mathrm{A}(\lambda z) > 0 \text{ on } \lambda \in (0,\lambda_1),$$
$$\frac{d}{d\lambda}\mathrm{A}(\lambda z) < 0 \text{ on } \lambda \in (\lambda_1,\infty) \text{ and}$$
$$\frac{d}{d\lambda}\mathrm{A}(\lambda z) = 0.$$

Since $\mathrm{A}(\lambda z)\big|_{\lambda=0} = \|\Delta z\|_m^m > 0$ and $\lim_{\lambda\to+\infty}\mathrm{A}(\lambda z) = -\infty,$ there exists

$$\lambda^* > 0 \text{ so that } \mathrm{A}(\lambda^* z) = 0,$$
$$\mathrm{A}(\lambda z) > 0 \text{ on } \lambda \in (0,\lambda^*) \text{ and}$$
$$\mathrm{A}(\lambda z) < 0 \text{ on } \lambda \in (\lambda^*,+\infty).$$

So,

$$\frac{d}{d\lambda}J(\lambda z) > 0, \text{ on } (0,\lambda^*),$$
$$\frac{d}{d\lambda}J(\lambda z) < 0 \text{ on } (\lambda^*,+\infty) \text{ and}$$
$$\frac{d}{d\lambda}J(\lambda z) = 0.$$

Therefore (ii) is valid.
(iii) From the definition of $I(z),$ we get

$$\begin{aligned} I(\lambda z) &= \lambda^m\|\Delta z\|_m^m - \lambda^r \ln\lambda\|\nabla z\|_r^r - \lambda^r\int_\Omega |z|^r \ln z dx, \\ &= \lambda\left[\lambda^{m-1}\|\Delta z\|_m^m - \lambda^{r-1}\ln\lambda\|\nabla z\|_r^r - \lambda^{r-1}\int_\Omega |z|^r \ln z dx\right] \\ &= \lambda\frac{d}{d\lambda}J(\lambda z), \end{aligned}$$

here $\lambda > 0.$ When combined with (ii), result (iii) holds.

Lemma 3 *Let* $z \in X$ *satisfy* $I(z) < 0.$ *Later, there exists a* $\lambda^* \in (0,1)$ *such that* $I(\lambda^* z) = 0.$

Proof For $\forall\lambda > 0,$ we get $I(z) = \|\Delta z\|_m^m - \int_\Omega |z|^r \ln z dx$

$$I(\lambda z) = \lambda^2\left[\|\Delta z\|^{m-2} - \varphi(\lambda)\right],$$

here

$$\varphi(\lambda) = \lambda^{r-2}\int_\Omega |z|^r \ln z dx + \lambda^{r-2}\ln\lambda\|\nabla z\|_r^r.$$

By $I(z) < 0,$ we obtain

$$\int_\Omega |z|^r \ln z dx > \|\Delta z\|_m^m.$$

By (33) and (35), we get

$$\varphi(1) = \int_\Omega |z|^r \ln z dx > \|\Delta z\|_m^m > 0,$$

$$\varphi(\lambda) = \lambda^{r-2}\int_\Omega |z|^r \ln z dx + \lambda^{r-2}\ln\lambda\|\nabla z\|_r^r \to 0^+.$$

Combining (33),(36) and from the above equation, we can deduce that there is

$\lambda^* \in (0,1)$ so that

$$\varphi(\lambda^*) = \|\Delta z\|_m^m$$

and $I(\lambda^* z) = 0.$ The proof is completed.

Lemma 4 *Suppose that (A1) and (A2) hold and $z(x,t)$ be a weak solution of problem (5). Then, $E(t)$ is nonincreasing function, that is*

$$E'(t) \le 0.$$

Proof Multiplying the equation (5) with z_t and integrating with respect to x over the domain Ω, we obtain

$$\left\| \frac{z_\tau}{|x|^{s/2}} \right\|^2 + \frac{1}{m}\frac{d}{dt}\|\Delta z\|^m = \int_\Omega |z|^{r-2} z z_t \ln z dx.$$

Through direct calculation, for the third term from the left it can be seen that
If similar operations are performed on the left side of the equation,

$$\int_\Omega |z|^{r-2} z z_t \ln z dx = -\frac{1}{r}\frac{d}{dt}\int_\Omega |z|^r \ln z dx - \frac{1}{r^2}\frac{d}{dt}\|z\|_r^r.$$

Inserting 42 into 39, we get

$$\left\| \frac{z_\tau}{|x|^{s/2}} \right\|^2 + \frac{d}{dt}\left[\frac{1}{m}\|\Delta z\|_m^m - \frac{1}{r}\int_\Omega |z|^r \ln z dx + \frac{1}{r^2}\|z\|_r^r \right] \le 0.$$

The proof is completed.

Lemma 5 *Assume that $z_0 \in X$. Later,*
(i) *the solution z of problem 5 with $z_0 \in W_1 \cup W_1^+$ satisfies that $z(t) \in W_1 \cup W_1^+$ for all $t \in [0, T^*]$.*
(ii) *the solution z of problem 5 with $z_0 \in W_1 \cup W_1^-$ satisfies that $z(t) \in W_1 \cup W_1^-$ for all $t \in [0, T^*]$.*

Proof (i) Suppose that $z(t)$ be the weak solution by problem 5 with $z_0 \in W_1 \cup W_1^+$, The meaning is that $J(z_0) < d$, $I(z_0) > 0$. The time variable on $(0,t)$ is integrated on both sides with respect to t 39, we have

$$J(z(t)) + \int_0^t \left\| \frac{z_\tau}{|x|^{s/2}} \right\|^2 d\tau = J(z_0).$$

By (44), we can get

$$J(z) < J(z_0) < d, \ \forall t \in [0, T^*].$$

Next, we assert that $I(z(t)) > 0$ for all $t \in [0, T^*]$, which, combined with equation (45), implies that $z(x,t) \in W_1 \cup W_2$. Otherwise, by the continuity of $I(z)$, there would exist a time $t_0 \in (0, T^*)$ such that $I(z(t)) > 0$ for $t \in [0, t_0)$ and $I(z(t_0)) = 0$ while $z(t_0) \ne 0$. This would imply that $z(t_0) \in \mathrm{N}$. Referring to the definition of d, it is evident that $d \le J(z(t_0))$ which leads to a contradiction with equation $d \le J(z(t_0))$.

Therefore, $z(t)\in W_1\cup W_1^+$ for all $t\in[0,T^*]$.

(ii) Since the proof is similar to part (i), we skip it.

Lemma 6 (*see [12]*) *(Hardy-Sobolev Inequality). Suppose that* $R^N : R^k\times R^{N-k}$, *and* $x=(y,z)\in R^k\times R^{N-k}$ *. For specific values of* γ *and* s *, there exists a range where* $1<\gamma<N$, $0\le s\le\gamma$, *and* $s<k$ *, such that* $m(s,N,\gamma)$ *equals* γ *, and the ratio* $\frac{N-s}{N-\gamma}$ *is constant. Additionally,* $H=H(s,N,\gamma,k)$ *is positive*

$$\int_{R^N}|y|^{-s}|z(x)|^m dx\le H\left(\int_{R^N}^{\gamma}|\Delta z|^{\gamma}dx\right)^{\frac{N-s}{N-\gamma}},\forall z\in W_0^{1,\gamma}(\Omega).$$

Remark 7 *When* $m=2$ *is set, the above inequality becomes*

$$\int_{\Omega}|x|^{-s}|z(x)|^2 dx\le H\left(\int_{\Omega}|\Delta z|^{\frac{2N}{N-s+2}}dx\right)^{\frac{N-s+2}{N}}.$$

From $0\le s\le 2$ *and* $N>2$ *, we can obtain by Hölder's inequality*

$$\begin{aligned}\int_{\Omega}|x|^{-s}|z(x)|^2 dx &\le H\left(\int_{\Omega}|\Delta z|^{\frac{2N}{N-s+2}}dx\right)^{\frac{N-s+2}{N}}\\ &\le H|\Omega|^{\frac{N-s+2}{N}-1}\|\Delta z\|^2\\ &= H_N\|\Delta z\|^2.\end{aligned}$$

We introduce the following inequality to address the logarithmic nonlinearity.

Lemma 8 *[11] Assume that* μ *is a positive number. Then we have the following inequalities:*

$$s^q\ln s\le(e\mu)^{-1}s^{q+\mu},\ \text{ for all } s\ge 1,$$

and

$$\left|s^q\ln s\right|\le(eq)^{-1},\ \text{ for all } 0<s<1.$$

Lemma 9 *[9]. Assume that* $f : R^+\to R^+$ *be a nonincreasing function and* σ *be a positive constant so that:*

$$\int_t^{+\infty}f^{1+\sigma}(s)ds\le\frac{1}{\omega}f^{\sigma}(0)f(t),\qquad\forall t\ge 0.$$

Then we get

(i) $f(t)\le f(0)e^{1-\omega t}$, *for all* $t\ge 0$, *whenever* $\sigma=0$.

(ii) $f(t)\le f(0)\left(\frac{1+\sigma}{1+\omega\sigma t}\right)^{\frac{1}{\sigma}}$, *for all* $t\ge 0$, *whenever* $\sigma>0$.

We demonstration from Theorem 10 that the norm $\|z\|_{H_0^2(\Omega)}$ decays exponentially to problem (5).

3. DECAY

In this part, we show the decay of weak solution to problem (5).

Theorem 10 *Assume that* $z(t)$ *be the solution to problem (5) and* m,r *satisfy if* $z_0 \in \mathrm{W}_1^+$, *then*

$$\|\Delta z\|^2 \le \|\Delta z_0\|_m^2 \left(\frac{m-1}{1+c_3(m-2)t}\right)^{\frac{1}{m-2}},\ t \ge 0.$$

here C_3 *is exist positive constant, such that* $m=2,$ *then there exist positive constants* C_4 ,

$$\|\Delta z\|^2 \le \|\Delta z_0\|_m^2 e^{\frac{1}{2}(1-C_4 t)},\ t \ge 0.$$

Proof We know from the results of $z_0 \in \mathrm{W}_1^+$ and global weak solutions that $z(t) \in \mathrm{W}_1^+$. From here, from (12), we get

$$\left(\frac{1}{m}-\frac{1}{r}\right)\|\Delta z\|_m^m + \frac{1}{r^2}\|z\|_r^r \le J(z(t)) \le J(z_0) < d.$$

By a direct calculation we arrive at the following conclusion:

$$\lambda_0\left[\left(\frac{1}{m}-\frac{1}{r}\right)\|\Delta z\|_m^m + \frac{1}{r^2}\|z\|_r^r\right] \ge J(\lambda^* z(t)) \ge d,$$

here $\lambda_0 = \max\left\{(\lambda^*)^m, (\lambda^*)^r\right\}$. Combining with (64), we obtain

$$\lambda_0 \ge \left\{\left(\frac{d}{J(z_0)}\right)^{\frac{1}{m}}, \left(\frac{d}{J(z_0)}\right)^{\frac{1}{r}}\right\} > 1,$$

so we can conclude that $\lambda^* > 1$, which means:

$$\lambda^* \ge \left(\frac{d}{J(z_0)}\right)^{\frac{1}{r}} > 1.$$

By (14), we get

$$\begin{aligned} 0 &= I(\lambda^* z) \\ &= (\lambda^*)^m \|\Delta z\|_m^m - (\lambda^*)^r \int_\Omega |z|^r \ln z dx - (\lambda^*)^r \ln(\lambda^*)\|z\|_r^r \\ &= (\lambda^*)^r I(z) - \left[(\lambda^*)^r - (\lambda^*)^m\right]\|\Delta z\|_m^m - (\lambda^*)^r \ln(\lambda^*)\|z\|_r^r \end{aligned}$$

In view of (70) and (73) we have*C*

$$\begin{aligned} I(z) &= \|z\|_r^r \ln(\lambda^*) + \left[1-(\lambda^*)^{m-r}\right]\|\Delta z\|_m^m \\ &\ge \mathrm{C}_1\|\Delta z\|_m^m, \end{aligned}$$

here $\mathrm{C}_1 = 1-\left(\frac{d}{J(z_0)}\right)^{1-\frac{r}{m}}, m < r.$

From equation (14) and Lemma 8, we get

$$\begin{aligned}
\int_t^T I(z)ds &= \int_t^T \left(\|\Delta z\|_m^m - \int_\Omega |z|^r \ln z dx \right) ds \\
&= -\frac{1}{2}\int_t^T \frac{d}{dt}\left\| \frac{z}{|x|} \right\|^2 ds \\
&= \frac{1}{2}\left\| \frac{z(t)}{|x|} \right\|^2 - \frac{1}{2}\left\| \frac{z(T)}{|x|} \right\|^2 \\
&\leq \frac{1}{2}\left\| \frac{z(t)}{|x|} \right\|^2 \\
&\leq C_H \|\Delta z(t)\|_m^2,
\end{aligned}$$

here $\mathrm{C}_4 = \frac{\mathbf{R}_n + \mathbf{C}}{2}$, C is the optimal embedding constant.
From (74) and (76), we obtain

$$\int_t^T \|\Delta z(s)\|_m^m ds \leq \frac{\mathrm{C}_H}{2\mathrm{C}_1} \|\Delta z(t)\|_m^2 = \frac{1}{\mathrm{C}_2} \|\Delta z(t)\|_m^2, \quad \text{for all } t \in [0, T],$$

assume that $T \to +\infty$ in (78), by virtue of Lemma 9, it follows that

$$\|\Delta z\|_m^2 \leq \|\Delta z_0\|_m^2 \left(\frac{m-1}{1 + c_3(m-2)t} \right)^{\frac{1}{m-2}}, \; t \geq 0.$$

The proof of Theorem 10 has been completed.
□

Author Contributions
For research articles with several authors, a short paragraph specifying their individual contributions must be provided. The following statements should be used "Conceptualization, A.F. and E.P.; methodology, A.F. and E.P.; software, A.F. and E.P.; writing—original draft preparation, A.F. and E.P.; writing—review and editing, A.F. and E.P.. All authors have read and agreed to the published version of the manuscript." Please turn to the CRediT taxonomy for the term explanation. Authorship must be limited to those who have contributed substantially to the work reported.

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